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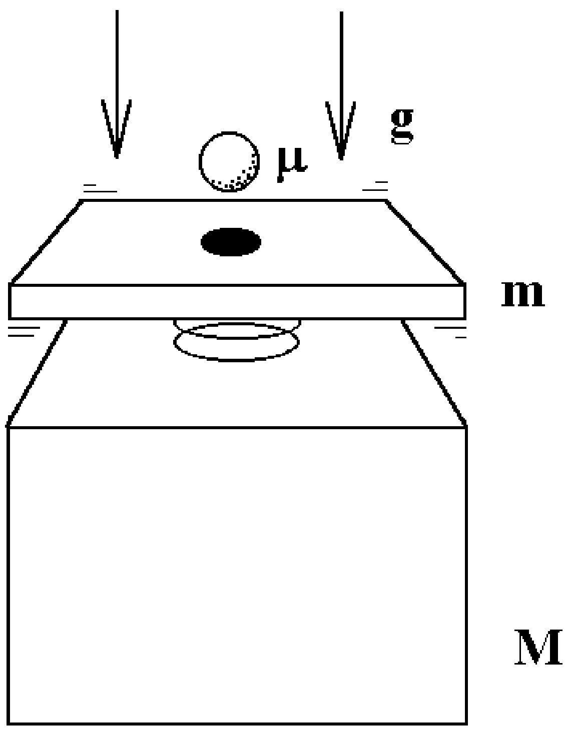
BOUNCING BALL WORKBENCH



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I. INTRODUCTION

A ball bouncing on a vibrating surface makes one of the simplest nonlinear dynamical systems.[1] Experimenting with this model allows one to observe many phenomena which appear in dynamical systems only in presence of a nonlinearity.[2] Laboratory experiments of this kind have been described by different authors.[3] In this paper we present a computer program which simulates a complete environment within which such experiments can be performed. The equations on which this simulation is based are realistic i.e. they do not allow any physically false solutions [4] and, on the other hand, they describe properly such a subtle phenomena as the mute or the chirping modes.[5]

The experiments we describe below are not a kind of a demo which the user should passively contemplate. What the BB Workbench provides is but a complete experimental environment - the experiments must be performed by the user himself/herself. Thus, one must set appropriate values to such parameters as the surface vibration amplitude, restitution factor of the ball-surface collisions, drop the ball on the surface from different heights, observe and analyse its motion. The program offers two modes in which the experiments are visualised:

(i) a realistic one, marked "B", in which a ball is actually seen to be bouncing on a vibrating surface, while parameters of its collisions are simultaneously marked within the Poincare map of the system,

(ii) a more symbolic one, marked "N", in which the trajectories of both the ball and the surface are presented in a time-height plot (the Poincare map is also seen here).

Plots of the latter kind are used below to illustrate consecutive experiments we suggest. Descriptions of the experiments contain also some general comments allowing the reader to get a better intuitive insight into the physics of the studied system.

Obviously, the experiments we describe are but a very limited sample of what can be actually done using the numerically simulated workbench. Having once entered the land of nonlinear phenomena the reader will certainly find his own ways to its most interesting places.

REFERENCES

[1]

For an excellent review of experimental systems within which chaotic modes of motion have been observed see:
"*Chaotic Vibrations*" by Francis C. Moon, John Wiley, New York 1987. Unfortunately, Fig. 3-5, p.76, aimed at illustrating chaotic behaviour of the bouncing ball is rather misleading. The trajectory presented in part (a) of the Fig. is qualitatively wrong. As easy to check, in its free fall motion the ball follows different pieces of always the same parabola - the drawing suggests a different rule. Part (b) of the same Fig. presents physically unrealistic solution of Eq. 3-2-9. The problem is that this simplified equation provides a reasonably good description of but periodic modes of the ball motion. It completely fails, however, in the chaotic regime and allows the ball to appear at the other side of the collision surface.

[2]

There is a considerable number of books and review papers which present the whole variety of phenomena appearing in nonlinear system. Let us mention here just a few classic texts:
"*Regular and Stochastic Motion*" by A.J. Lichtenberg and M.A. Lieberman, Springer-Verlag, New York, 1983;
"*Deterministic Chaos*" by H.G. Schuster, Physik-Verlag, Weinheim, 1984 and "*L'Ordre dans le Chaos*" by P. Berge, Y. Pomeau and Ch. Vidal, Hermann, Paris, 1985.
For a collection of original papers see:
"*Chaos*", ed. Hao Bai Lin, World Scientific, Singapore, 1984.
French readers are strongly advised to read the excellent introduction:
"*Chaos et Determinisme*" by Vincent Croquette in **Pour la Science**, Decembere 1982.

[3]

The BB model can be seen as a particular modification of a model introduced by Enrico Fermi: in absence of gravity a ball bounces between two parallel walls of which one is vibrating
E. Fermi, Phys.Rev. 75, 1169 (1949).
First numerical simulations of this system were performed by Stanisław Ulam
S. Ulam, Proceedings of the 4th Berkeley Symp. on Math. Stat. and Probability, University of California Press, 3, 315 (1961).
For an extensive bibliography of papers which followed see Lichtenberg and Lieberman.
First papers which dealt with the BB model in its "gravity bounded" version were theoretical and came from the Russian school of Zaslavsky. For references see: "*Statistical Irreversibility in Nonlinear Systems*" by G.M.Zaslavsky, Moscow, 1970 (in Russian !).
The next one seems to be:
P.J. Holmes, J.Sound Vib., 84, 173 (1982).
First experimental work on the BB model was reported in:

P. Pierański, J.Phys. (Paris), 44, 573 (1983).
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P. Pierański and R.Bartolino, J.Phys. (Paris) 46, 687 (1985);
M. Franaszek and P. Pierański, Can.J.Phys. 63, 488 (1985).
Also next authors describing the experimental realisations of the BB model re-discover it on their own:
N.B. Tufillaro et all. J.Phys (Paris) 47, 1477 (1986);
S. Celasschi and R.L. Zimmerman, Phys.Lett. 120A, 447 (1987).
For description of more sophisticated phenomena predicted and found in the system see also:
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K. Wiesenfeld and N. Tufillaro, Physica (Amsterdam) 19D, 355 (1986);
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A. Mehta and J.M. Luck, Phys.Rev.Lett. 65, 393 (1990).

[4] R.M. Everson, Physica (Amsterdam) 19D, 355 (1986),

[5] Z. Kowalik et all. Phys.Rev. 37A, 416 (1988),

II. THE BOUNCING BALL MODEL

A ball is dropped onto a flat, strictly horizontal surface attached to the membrane of a vibrator (e.g. loudspeaker). The frequency of the surface vibration is kept constant; its amplitude A is controlled by the experimenter. See Fig.**Error! Bookmark not defined.**.

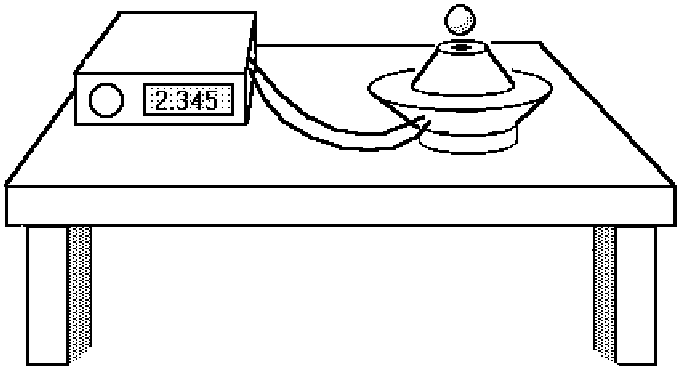

Fig. 1 Experimental set-up of the bouncing ball model.

We assume that the bounces of the ball are strictly vertical (what in real experiments needs some precautions since the ball easily enters a sideways oscillations) and that the amplitude and form of the vibration of the surface remain intact no matter which mode of its motion the ball enters. Under such assumptions the BB model can be considered as a particular three body system. See Fig. 2. Three masses, M » m » μ, move along a line interacting with each other via forces of different nature.

The big mass M (the laboratory within which the body of the vibrator is fixed) is coupled to the medium mass m (collision surface fixed) via an ideal spring, what makes this subsystem linear (harmonic).

The tiny ball μ is attracted to M via gravitational forces but on its free fall to the latter it meets the collision surface with which its interaction is strongly repulsive and of extremely short range. Thus, as a whole the μ-M subsystem is strongly nonlinear.

The theoretical BB model can be considered also in its conservative version but for obvious reasons practical realisation of the latter is impossible since in each collision between masses μ and m a part of their kinetic energy will be lost. We assume that in the

collision surface reference frame any collision has a similar scenario:

(i) the ball arrives to the surface with velocity $w-$

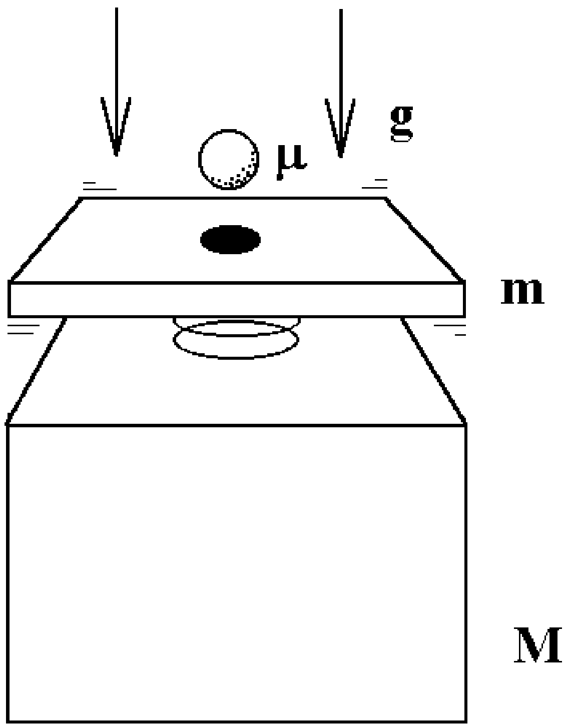

Fig. 2 Bouncing ball model seen as a three body system.

(ii)during the collision a part of the kinetic energy is dissipated
(iii) the ball leaves the surface with velocity

$$w^{+} = -kw^{-} \tag{1}$$

where $0 < k < 1$.
The above equation and the assumption that the collision surface moves harmonically make the starting point for the derivation of the equations of motion we present in the next section.

III. EQUATIONS OF MOTION

To simplify notation and make the analysis that follows as general as possible, we introduce dimensionless length and time variables.
Thus, we assume that:
dimensionless time

$$\Theta = \frac{2\pi t}{T_s} \tag{2}$$

dimensionless length

$$h = \frac{8\pi^2 l}{gT_s^2} \tag{3}$$

where T_S is the period of the surface vibration and g is the gravitational acceleration constant.

Constant factors (2π and $8\pi^2$) which appear in the definitions are introduced to obtain compatibility of the equation of motion with those of the standard map.

In terms of the dimensionless time and length the surface vibration period equals 2π and the gravitational acceleration is equal 2.

A formal derivation of the equations of motion is rather tedious, thus we rather directly present them and then discuss their physical meaning:

$$-A\ \cos(\Theta_i) + v_i\tau_i - \tau_i^2 = -A\ \cos(\Theta_i + \tau_i) \tag{4}$$

$$\Theta_{i+1} = \Theta_i + \tau_i \tag{5}$$

$$v_{i+1} = -k\ (v_i - 2\tau_i) + A(1+k)\ \sin(\Theta_{i+1}) \tag{6}$$

where v_i denotes the (dimensionless) velocity of the ball just after i-th collision and τ_i is the time length of the following jump.

Equations 3,4,5 describe the complete story of single jump.

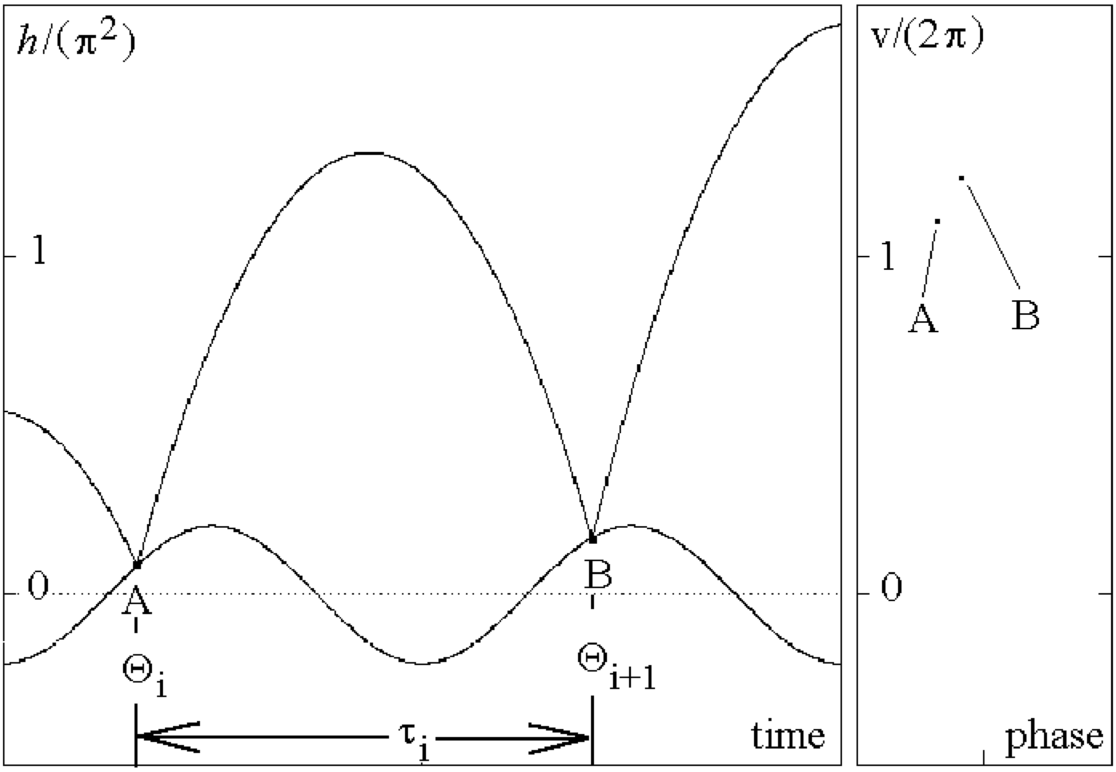

Fig. 3 Details of the ball motion.

It starts at time Θ_i at which i-th collision takes place. The ball emerges from the collision with velocity v_i. The collision takes place a height $[-A\cos(\Theta_i)]$ at which surface is found at this moment. The following trajectory of the ball is just a free fall within the gravitational field whose (dimensionless) acceleration equals 2. The left side of Eq. 3 describes the trajectory in terms of the local time τ. The right side describes simultaneous trajectory of the vibrating surface. When those two meet, after time τ_i, next i.e. (i+1)-th collision takes place. Equation 4 just gives its time. What has to be found to allow one to repeat the whole procedure is the velocity v_{i+1} of the ball just after the (i+1)-th collision. Let us do it.

After its free fall started with velocity v_i ball arrives to the (i+1)-th collision with velocity

$$(v_i - 2\tau_i) \tag{7}$$

as seen in the laboratory reference frame. (Factor 2 appearing within this formula is just the value of the acceleration constant). Thus, in the surface reference frame, within which Eq.1 was written, it equals

$$[(v_i - 2t_i) - A \sin(\Theta_{i+1})] \tag{8}$$

where A $\sin(\Theta_{i+1})$ term describes velocity of the surface as seen within the laboratory frame.

Using Eq.1. we may now find the velocity of the ball just after the collision (still in the surface reference frame) :

$$-k[(v_i - 2\tau_i) - A\sin(\Theta_{i+1})] \tag{9}$$

Thus, returning to the laboratory frame we finally find Eq.5 :

$$\begin{aligned} &v_{i+1} = \\ &-k[(v_i - 2\tau_i) - A\sin(\Theta_{i+1})] + A\sin(\Theta_{i+1}) = \\ &-k(v_i - 2\tau_i) + A(1-k)\sin(\Theta_{i+1}) \end{aligned} \tag{10}$$

Equations 3,4,5 describe properly most phenomena one observes in a laboratory BB model. However, their use in a simulation program needs caution. The problem is, that in addition to the audible "bouncing" modes of motion the ball may easily enter a particular "mute" mode, at which it just moves together with the surface - sitting on it. The BB Workbench simulation program takes into account such a possibility and reacts accordingly when it occurs, i.e. allows the ball to move in unison with the surface, controlling continuously the value of its acceleration. When the latter exceeds acceleration the gravitational field, the program makes the ball to leave the surface.

Note that Eqs. 3,4,5 are not differential equations and they determine details of the ball's motion only at the moments of collisions. Since, however, the rest of the ball's trajectory is a trivial free fall, this is in fact what one needs. Consequently, to present any mode of the ball's motion one does not need to present it all - it is sufficient to present only the phase $\Theta_i = \Theta_i \bmod 2\pi$ of each collision i.e. its position within the surface vibration period, and the velocity v_i just after it. The plot, within which points representing the two values are presented in the simulation, we shall further refer to as the "Poincare map" or "Poincare section".

For simple periodic modes in which the ball makes identical jumps and is colliding with the surface always at the same phase of its motion, Eqs. 4. can be simplified. Note, namely, that is this case the $A\cos(\Theta_i + \tau_i)$ and $A\cos(\Theta_i + \tau_i)$ terms of Eq.4a are equal, i.e. can be reduced. Consequently, Eq. 4a turns into but a simple equality

$$\tau_i = v_i \tag{11}$$

what allows one to write Eqs. 4 or 5as:

$$\Theta_{i+1} = \Theta_i + v_i \tag{12}$$

$$v_{i+1} = kv_i + l\sin(\Theta_i) \tag{13}$$

where

$$l = (1+k)\,A \tag{14}$$

These simplified equations of motion are very often regarded as "the equations of motion of the bouncing ball" which is quite misleading, since while describing properly its simplest periodic modes of motion they fail completely in the chaotic regime and allow solutions which cannot be realised in nature.

For instance, one can easily check that values $\Theta_i = 0$, $v_i = 0$ are a solution of Eqs. 12 and 13. On the other hand, it is also easy to see they do not describe any physically sensible mode of the ball's motion.

While not hesitating to use the equations in situations, where as a good approximation they can be used, one should be very cautious to apply them in situations where their applicability is not completely clear.

Obviously, there are other models for which Eqs. 12 and 13 are exact. For instance, at k=1 they describe equilibrium configurations of the chain of elastically coupled pendula. Here, Θ_i has the meaning of the angle which i-th pendulum makes with the direction of the gravitational field while v_i is simply the angle between the (i+1)-th and i-th pendula. For such a model the $\Theta_i = 0$, $v_i = 0$ solution is perfectly valid since it describes the simplest possible configuration in which all pendula just hang down. it is quite instructive to use this analogy in analysis of Exp.15.

Remark

There is a problem.

Equations 3,4 and 5 provide a precise description of the bouncing ball in any of its "bouncing modes":

(i) equation 3 allows one to find the time length τ_i of the i-th jump, which started at a known time Θ_i, when the i-th collision took place, with the known initial velocity v_i,

(ii) as equation 4 indicates, adding the value of the time length τ_i to the time Θ_i determines the time Θ_{i+1} of the next collision,

(iii) equation 5 tells, how to find the velocity v_{i+1} with which the ball will start to the next jump.

And so on.

What, unfortunately, is not true, is the implicit assumption that the ball never ceases to bounce. As we shall see, in reality the ball may enter a such a sequence of bounces which converges in a finite time to the surface and makes the ball "sit" peacefully on it. From this moment on the evolution of the ball motion is no more described by Eqs. 3,4,5 - the ball simply moves together with the surface until something (e.g. excessive acceleration of the surface itself) or somebody (the reader hitting an appropriate key) will make it to leave the latter.

To deal with this problem, any realistic simulation algorithm must be able to detect when such a convergent sequence of bounces occurs, find its limit, and allow the ball move with the surface until it is forced to enter a new jump. The Bouncing Ball Workbench algorithm is realistic. We should be grateful for any remarks concerning its performance.

IV. EXPERIMENTS

In what follows, we describe 15 experiments which can be easily performed by the reader within the laboratory simulated by the BB Workbench program.

The experiments are arranged in a sequence which in principle should be followed. Descriptions of the preparation procedures are initially detailed, but gradually, as the experimenter becomes acquainted with the simulated laboratory environment, they become brief and laconic. Each experiment is illustrated with Fig.s showing phenomena to be observed and a brief explanation of their physical meaning. Some experiments end with a list of problems, whose solutions may shed more light on what has been observed in the particular experiments. Solving them is advisable but not necessary.

Keys listed in Appendix A control a number of parameters used by the simulation procedure. One of the most essential is the time increment *dt* with which trajectories of the ball and the surface are calculated. The reader should learn to adjust it to a proper level (possibly small). One must remember that this discrete time simulation procedure is a kind of a dynamical system on its own and may produce solutions with have nothing in common with the motion of a real ball. The experimenter is also encouraged to use the keys which control scales of the plots presented by the program. Many phenomena can be well seen but at a proper choice of the scales.

As mentioned in the introduction, motion of the bouncing ball is presented within the simulation program in two different ways: a more realistic one, in which both the ball and the vibrating surface are actually seen moving, and a symbolic one, in which height-time plots of the ball and surface motion are continuously drawn. Switching between the two modes often helps to grasp a proper intuitive insight into the behaviour of the system.

The "." (full stop) key allows one to freeze execution of the simulation program. It has been introduced to facilitate hard-copying the phenomena recorded within the height-time plots. Any resident program of the PIZZAZ type can be used for this purpose.

EXPERIMENT 1 *Ball bouncing on a motionless surface*

1. Set the surface vibration amplitude A to 0 (Keys A,S).
2. Set the restitution factor k to an experimentally realistic value e.g 0.85 (Keys K,L).
3. Clear the Poincare map (Key P).
4. Drop the ball (Keys 1...5) several times from different heights and analyse its motion. Listen to the sound produced by the sequence of collisions. Note what happens when the restitution parameter k tends to 0 or 1.

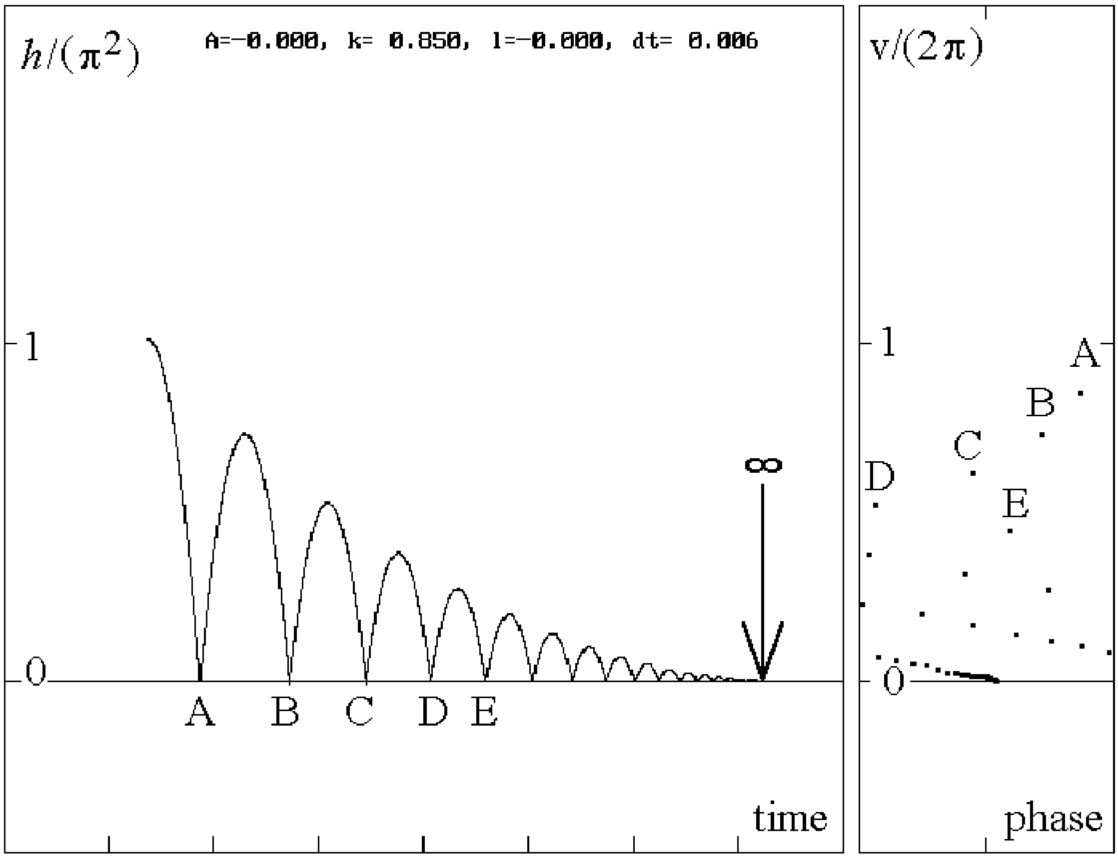

Fig. 4 Convergent sequence of bounces which occurs when the ball is dropped on a motionless surface. The number of bounces in the sequence is infinite but still its time length is finite.

Due to the dissipative nature of the ball-surface collisions consecutive bounces of the ball are shorter and shorter. The time intervals τ_i, $i = 1, 2, 3, ...$ between the consecutive collisions make a sequence which converges geometrically to zero:

$$\tau_{i+1} = k\tau_i, \; i = 1, 2, 3, ..., \text{ where } 0 \le k < 1 \qquad (15).$$

As a result, the total time length of the sequence of bounces is finite and equals:

$$\Theta_{\infty} - \Theta_1 = \sum_{i=1}^{\infty} \tau_i = \frac{\tau_1}{1-k} \tag{16}$$

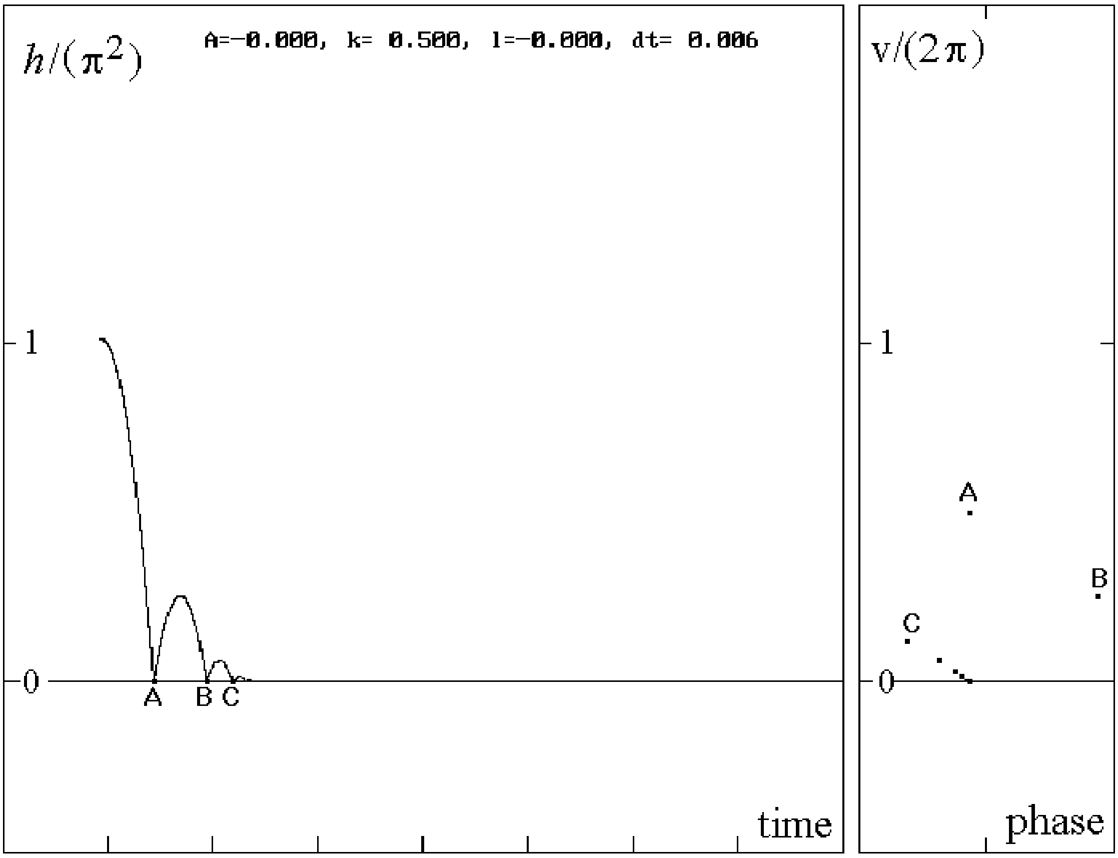

Fig. 5 For a lower value of the restitution parameter k the total time length of the convergent sequence of bounces becomes shorter.

PROBLEMS

1. Analyse observed motions in terms of frequency.
2. Find the qualitative difference between the motion of the damped harmonic oscillator (typical linear system) and that of the bouncing ball (typical nonlinear system).
3. Suggest an experimental procedure by which restitution factor *k* could be determined for a Ping-Pong ball bouncing on a table.

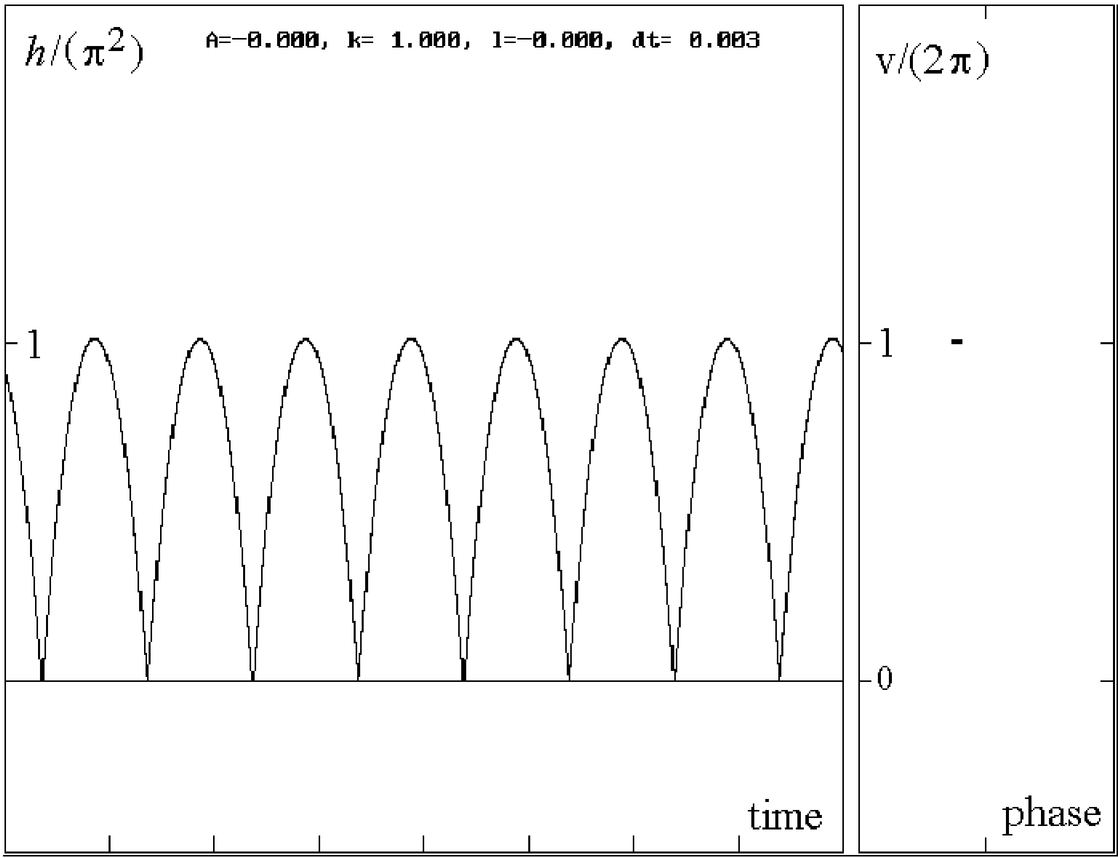

Fig. 6 When *k* is equal to 1, the limit of the conservative bouncing ball is reached. Since no energy is dissipated, all bounces are of the same height. There are no limits to the latter.

EXPERIMENT 2 *When A is too small.*

1. Set the restitution factor to a value $k < 1$ e.g. 0.85.
2. Set the surface vibration amplitude to a value $A < A^*$, where $A^* = \min(A^{(0)}_{MAX}, A^{(1)}_{MIN})$, while $A^{(0)}_{MAX} = g = 2$ and $A^{(1)}_{MIN} = 2\pi(1-k)/(1+k)$. For $k = 0.85$, $A^* = 0.51$.
3. Drop the ball several times from different heights and observe its motion. 4. Drop the ball many times from the same height and analyse the distribution of points marked by resulting trajectories within the Poincare map.

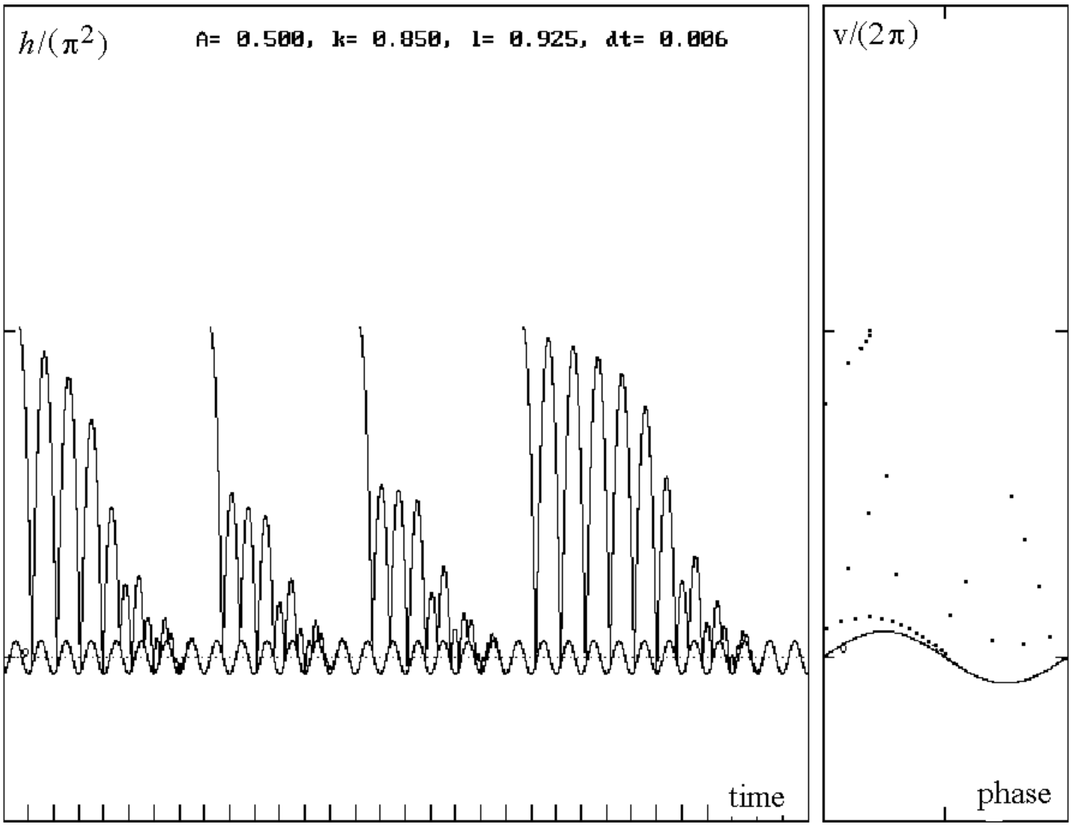

Fig. 7 Transients of motion observed when the ball is dropped onto the surface vibrating with too small amplitude. The set of points seen within the Poincare map comes from the last transient.

When the surface vibration amplitude is too small, the motion of the ball cannot be sustained no matter from what height and at which phase of the surface motion the ball is dropped. After an initial transient of bounces the ball enters a convergent sequence similar to that observed at A=0. As a result, in a finite time (although infinite number of bounces) the ball comes to the permanent contact with the vibrating surface and starts moving with it. This mute (since it produces no sound) mode of the ball motion we denote by $M^{(0)}$. Its image within the Poincare map differs qualitatively from images of any of the "bouncing" modes. Since the map is defined as such a section of the phase space of the ball-surface-laboratory system at which position of the ball is equal to that of the surface, the mute mode appears in it as a continuous, sine shaped line, which makes in practice the lower border for motion of phase points - never any trajectory may appear below it. The Fig. below demonstrates it clearly.

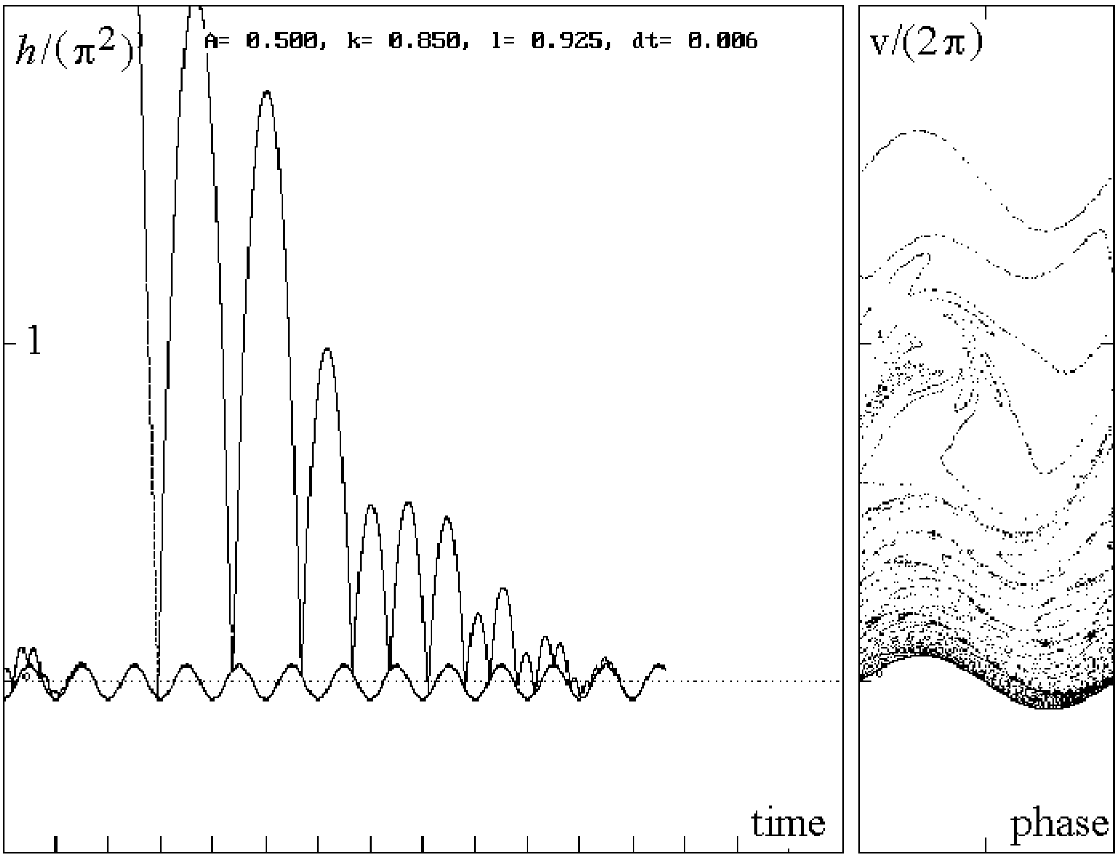

Fig. 8 The distribution of trajectory points which appears within the Poincare map when the ball is dropped many times from the same height.

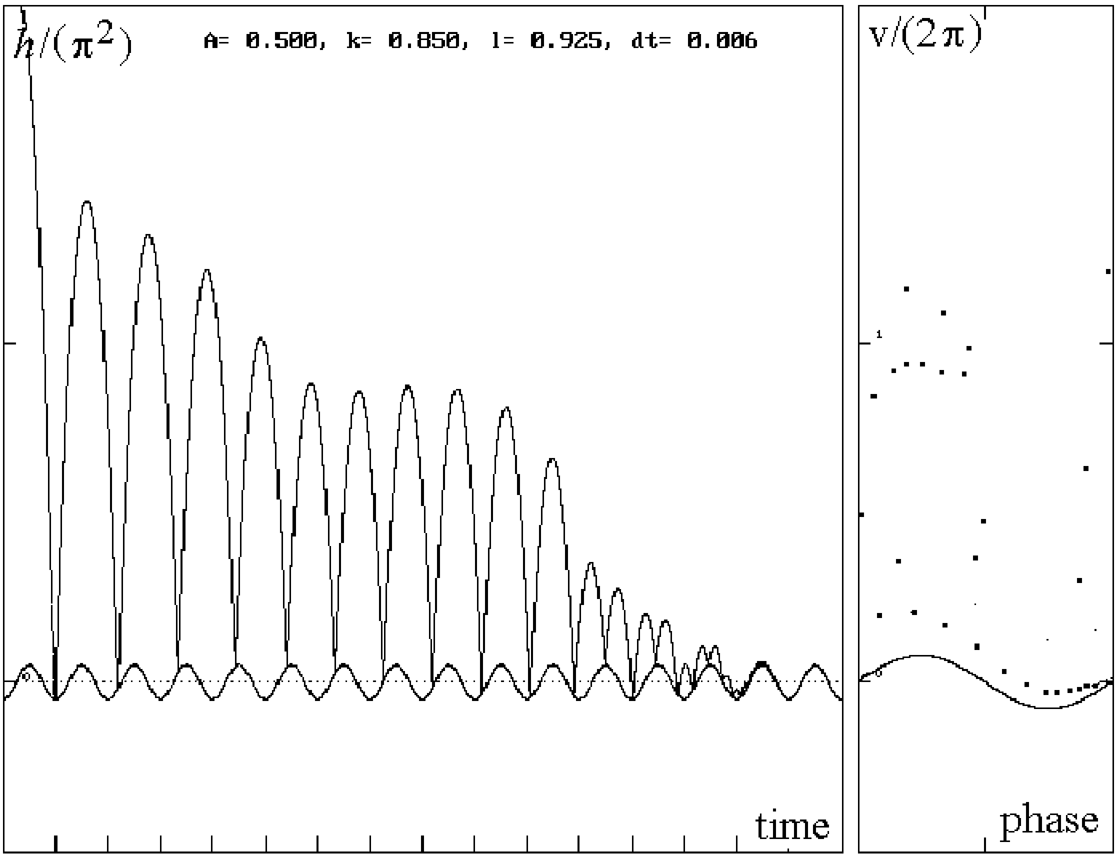

Fig. 9 A transient within which the motion of the ball was "almost sustained". Note the particular shape of the trajectory within the Poincare map. Look at the same region of the map in Fig. 7.

PROBLEMS

1. As the ball enters the $M^{(0)}$ mode "sitting down" onto the surface, try to increase A to such a value at which it starts loosing the permanent contact with the surface. Think about the physical meaning of this threshold.
2. Find an analytical derivation of the formulae given in the
Step 2.

EXPERIMENT 3 *First success - the ball enters $M^{(1)}$ mode.*

1. Set k<1 e.g. 0.85
2. Set A to a value greater then $A^{(1)}{}_{MIN}$. (See Step 2 of Experiment 2.) For k=0.85, $A^{(1)}{}_{MIN}$=0.51 thus A can be equal e.g. 0.65.
3. Drop the ball several times from different heights. Observe possible stable motion which may appear after the initial transients.

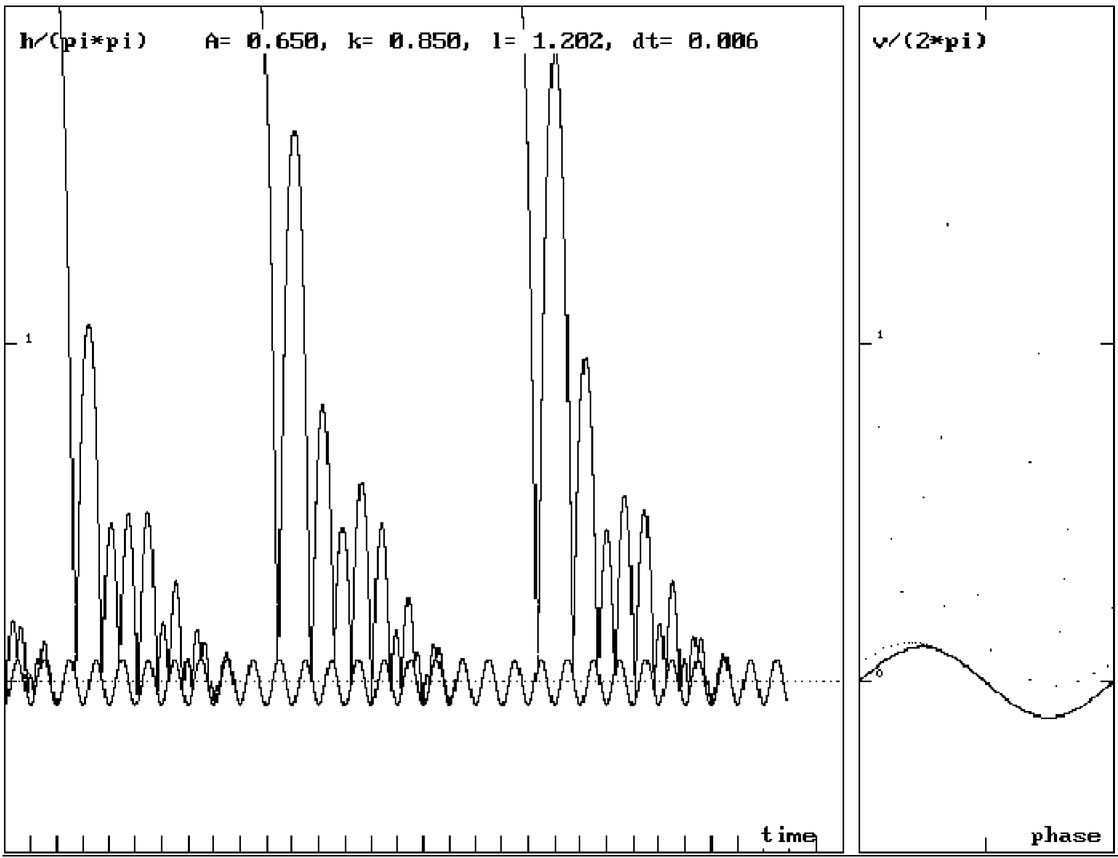

Fig. 10 The amplitude of the surface vibration has been increased above the $A^{(1)}{}_{MIN}$ threshold. In spite of that not every try ends with a success.

Depending on the height and phase at which the ball is dropped there are three possible outcomes of the experiment:

a) the ball enters a convergent sequence of bounces which as in the previous experiment leads either to the mute $M^{(0)}$ mode or, at small values of k, to a partially mute "chirping mode" $M^{(0)}{}_1$, (See Experiment 10)

b) the initial transient of bounces leads to a regular motion in which the time length of each bounce is equal to the period of the surface vibration. This mode of motion we denote $M^{(1)}$.

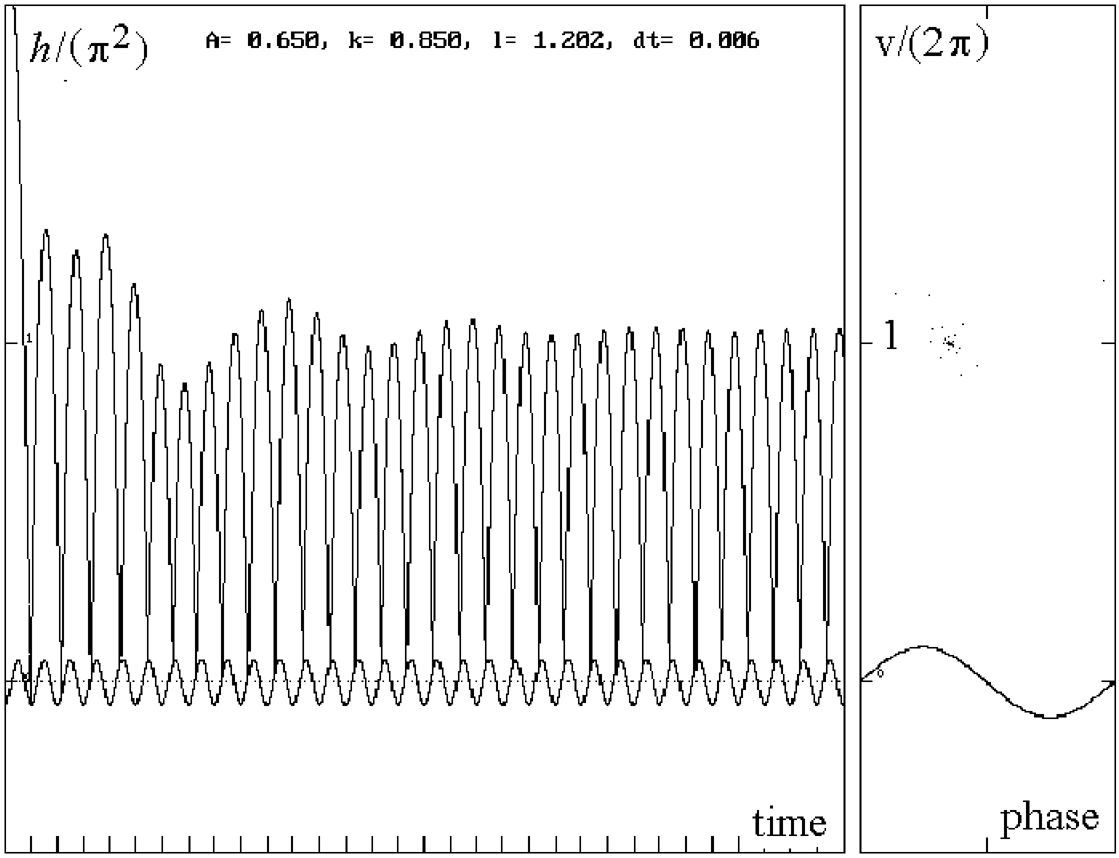

Fig. 11 This time the experiment ends successfully - after an initial hesitation the ball enters its simplest periodic mode of motion $M^{(1)}$.

PROBLEMS

1. Note the low frequency damped oscillation via which the ball converges to its final periodic motion. Study the dependence of its frequency on the distance $\Delta A=A\text{-}A^{(1)}{}_{\mathrm{MIN}}$.
2. Find the approximate differential equation which describes the oscillation. Is it linear?

EXPERIMENT 4 *Mode $M^{(1)}$ copes with varying A*

1. Using the procedure described in the previous experiment put the ball into its $M^{(1)}$ mode and bring the surface vibration amplitude A close to the $A^{(1)}_{MIN}$ threshold.
2. Increase slowly the amplitude of the surface vibration and observe what is happening to the ball motion. Does the ball start bouncing higher? Note the phase of the ball-surface collisions and the slope of the of the surface motion plot.

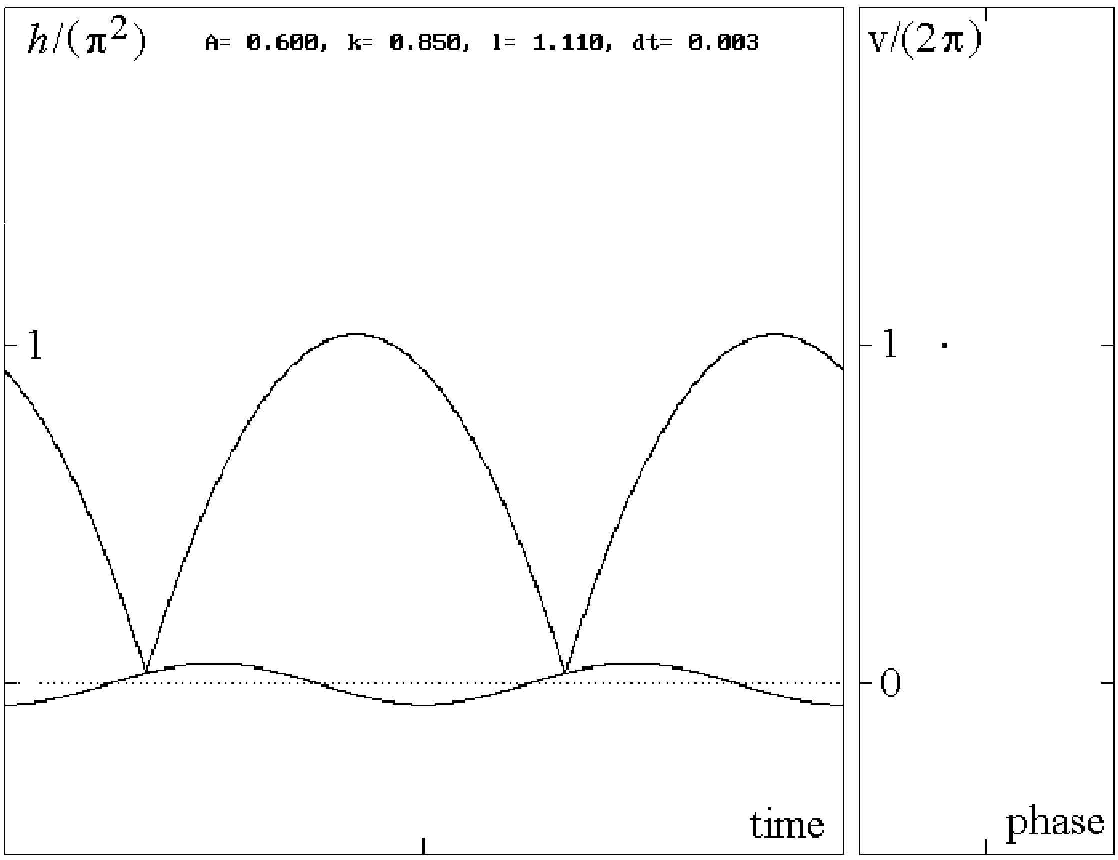

Fig. 12 For the surface vibration amplitude just above the $A^{(1)}_{MIN}$ threshold, the ball-surface collisions are located on the positive slope of the surface motion -cosine function.

As easy to note, in spite of the increasing amplitude of the surface vibration the height of the ball jumps remains constant. (The ball cannot simply start jumping higher since this would mean longer flights which would not match the periods of the surface vibration). What is happening instead is that, preserving its relative height, the sequence of jumps shifts as a whole searching within the surface vibration period such a position within the surface velocity would be the same as before.

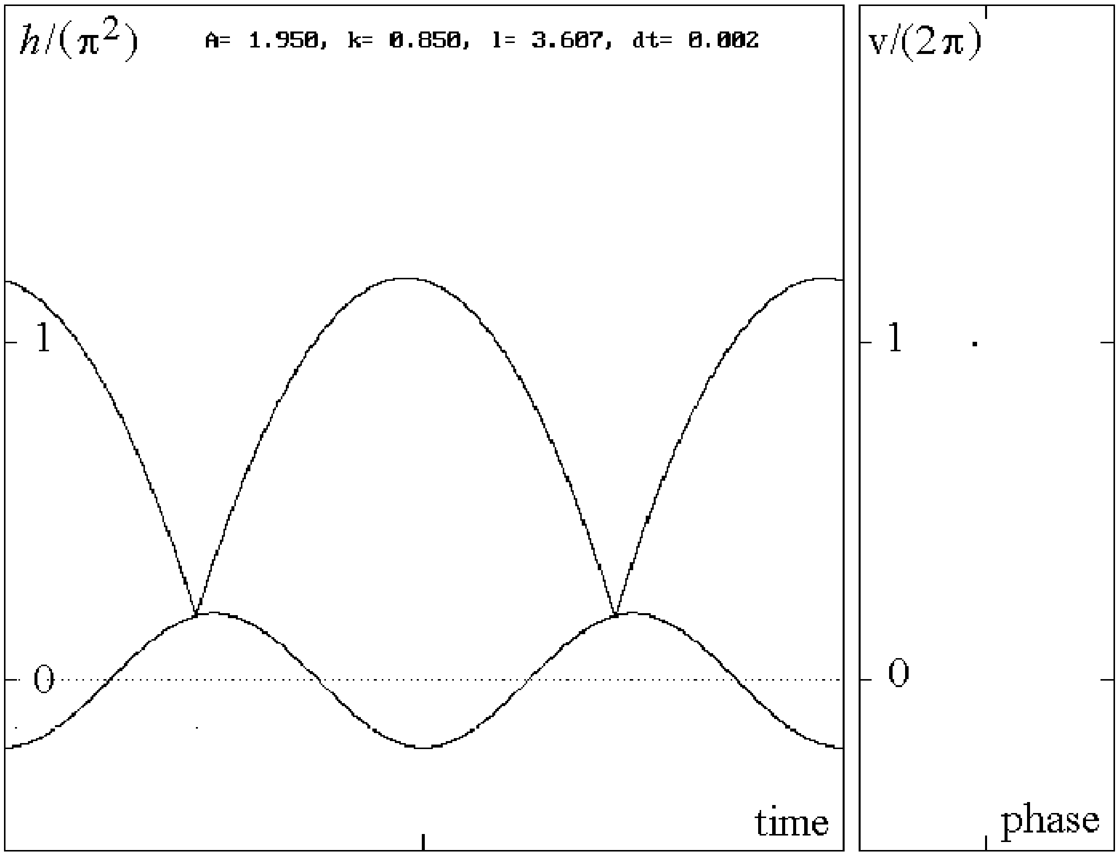

Fig. 13 $M^{(1)}$ mode at a larger value of the surface vibration amplitude. Compare the slope of the surface motion at the moments of collisions with that seen in Fig.11.

PROBLEMS:

1. Analyse the collision events within the $M^{(1)}$ mode. Prove the law

$$\Theta^{(1)}{}_0 = \pi - \varepsilon, \text{ where} \tag{17}$$

$$\varepsilon = \arcsin\left(2\pi \frac{1-k}{1+k} A^{-1}\right) \tag{18}$$

which describes position of the collisions within the surface vibration period.
2. Which is the maximum value for ε? Find its relation to the $A^{(1)}{}_{\mathrm{MIN}}$?
3. Try to observe in which manner the $M^{(1)}$ mode responds to perturbations, in particular in the vicinity of the lower and upper stability limits.

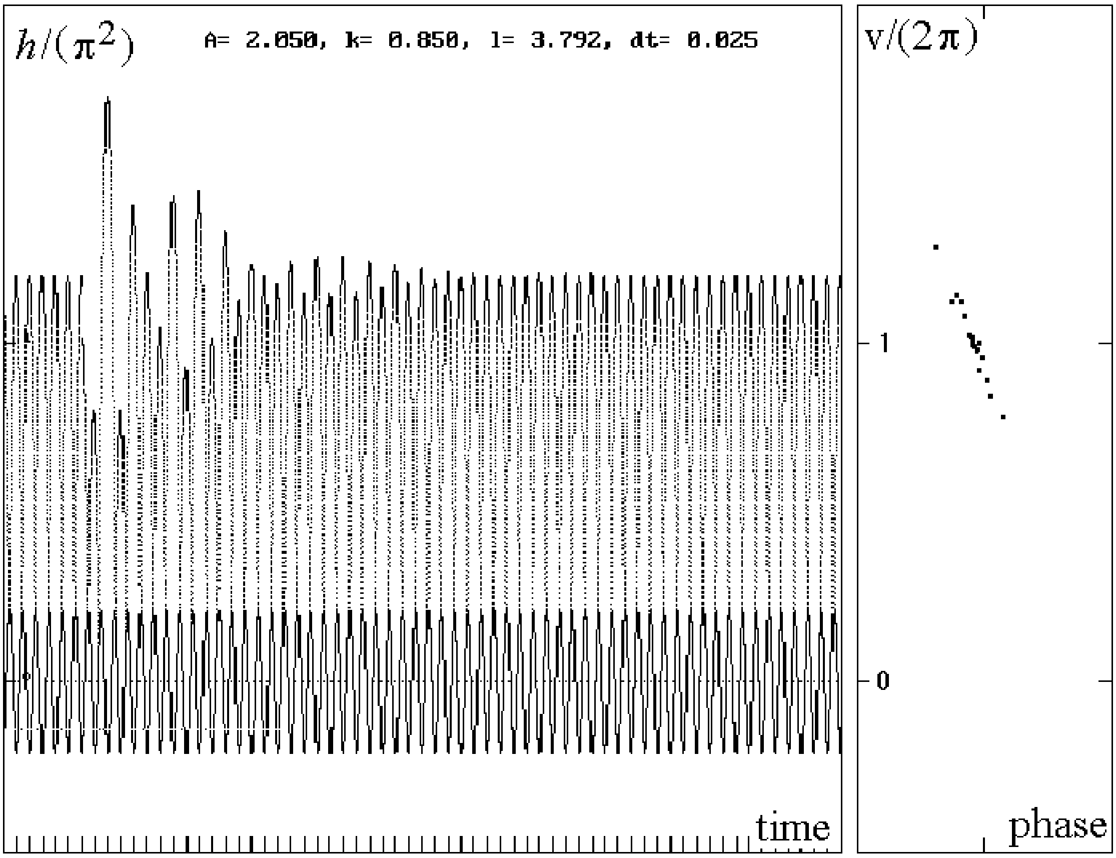

Fig. 14 Close to the upper stability limit the $M^{(1)}$ mode responds to a perturbation in a characteristic manner. Have a look at the envelope of the subsequences composed of every second jump.

Experiment recorded in the Fig. above indicates well, what happens when A approaches its upper stability limit. Apparently, when observed in a stroboscopic manner (every second period of the surface vibration), the response of the $M^{(1)}$ mode to a perturbation can be analysed in terms of a linear (damped) oscillator, whose characteristic frequency tends to zero.

<u>EXPERIMENT</u> 5 *Mode $M^{(1)}$ doubles its period.*

1. Set a value of k in the interval (0, 1).
2.. Set the surface vibration amplitude A to value above $A^{(1)}_{\mathrm{MIN}} = 2\pi(1 - k)/(1 + k)$.
3. Put the ball into the $M^{(1)}$ mode.
4. Increasing slowly A find the upper threshold $A^{(1)}_{\mathrm{MAX}}$ above which the $M^{(1)}$ mode looses stability.
5. Determine the period of the $M^{(1,\,2)}$ mode into which the $M^{(1)}$ mode is transformed.

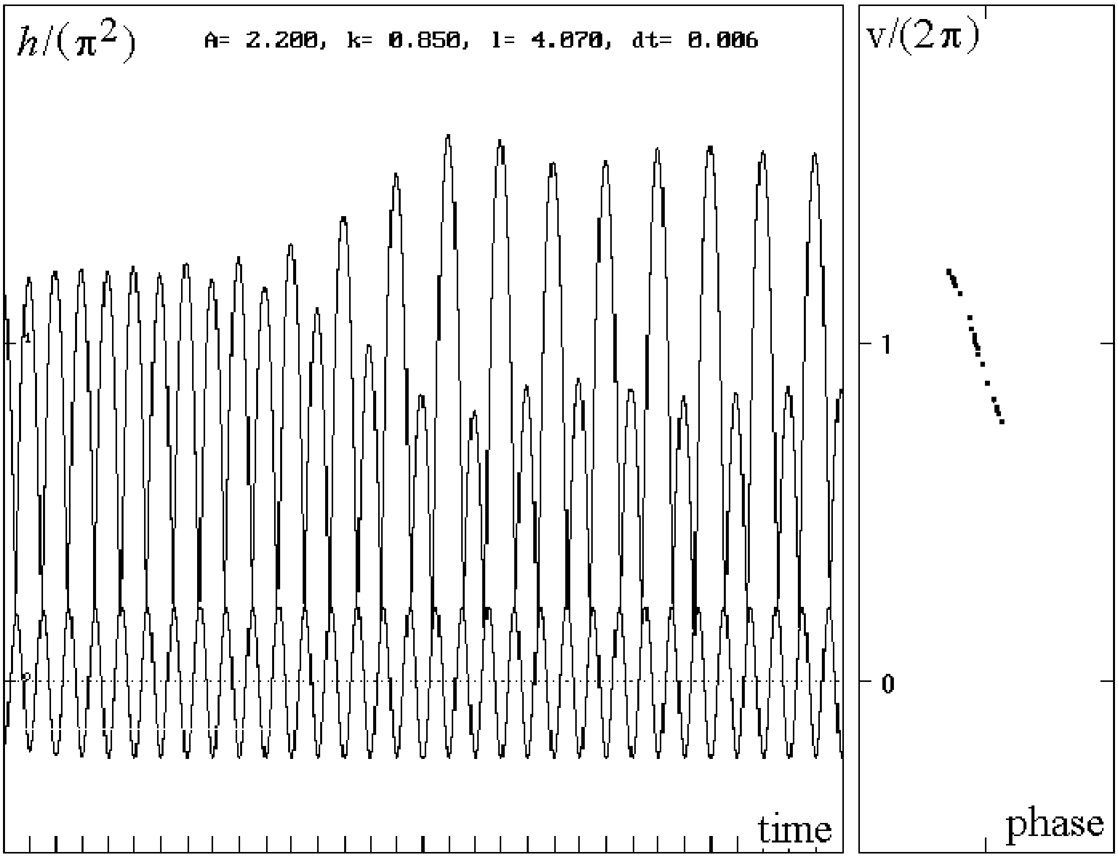

Fig. 15 Above a certain threshold, the $M^{(1)}$ mode looses stability and turns into its period doubled version $M^{(1,2)}$.

The transition from the $M^{(1)}$ to $M^{(1,\,2)}$ mode is called the <u>period doubling bifurcation</u>. In its vicinity a number of interesting phenomena occur. One can easily note, for instance, that close to the point the relaxation time of the system motion increases considerably. This, as it is called, critical slowing down phenomenon is typical to transitions in which one mode of motion gives, in a smooth manner, place to another to mode.

There are two types of collisions within the $M^{(1,\,2)}$ mode.

(1) at which the ball meets the surface at its going-up phase,

(2) at which the collision occurs while the surface is moving down.

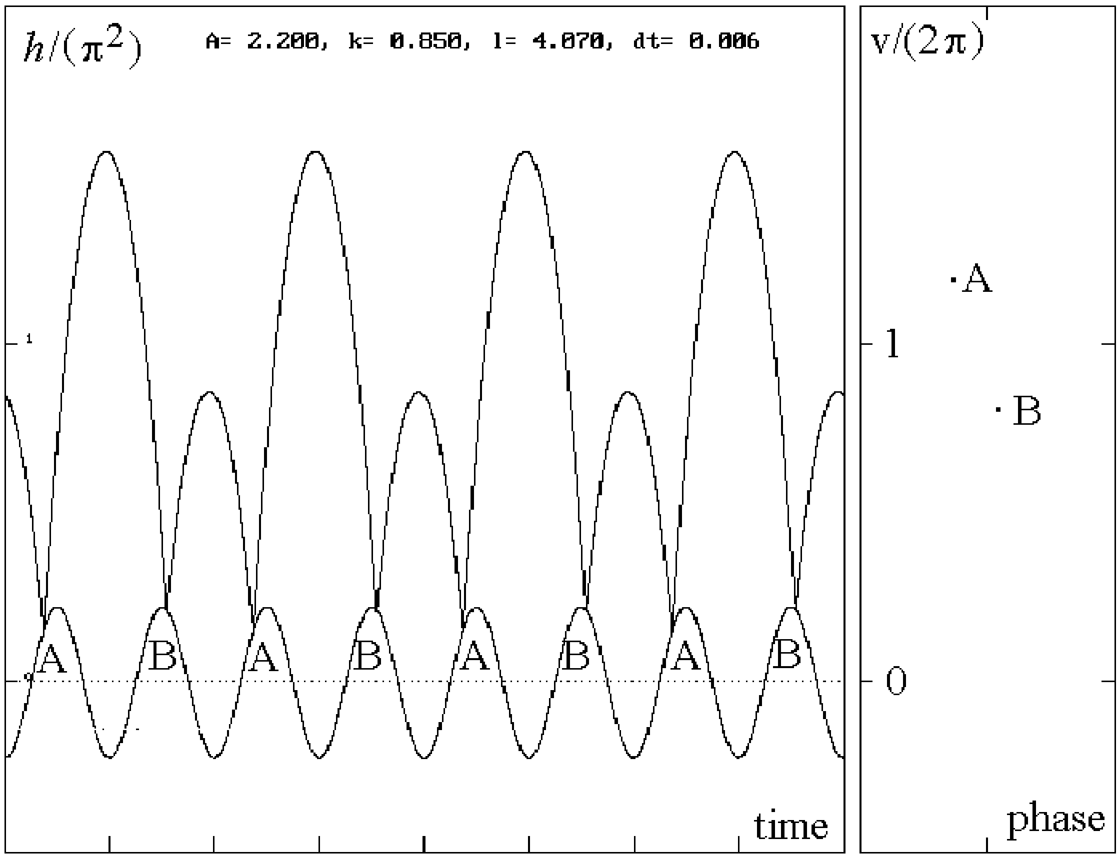

Fig. 16 Details of the ball motion within the $M^{(1,2)}$ mode.

One can ask, of course, why at all the $M^{(1)}$ mode turns into the $M^{(1,\,2)}$ mode. To answer this question one must take into consideration not only the problem of finding such a point within the surface vibration period at which the velocity of the up-moving surface is just right to sustain the $M^{(1)}$ mode.

Looking at the formula

$$\Theta^{(1)}{}_C = \pi - \varepsilon,$$

where $\varepsilon = \arcsin\left(2\pi \frac{1-k}{1+k} A^{-1}\right)$,

which describes such a point one can see that it has but a lower (at which $\varepsilon = \pi/2$) but no upper limit. Thus, the $M^{(1)}$ mode exists for any value $A > A^{(1)}{}_{\mathrm{MIN}}$. The problem is that above a certain $A^{(1)}{}_{\mathrm{MAX}}$ threshold it looses its stability i.e. when perturbed it does not return to the previous shape but goes away from it. To understand why is that so one must consider just the problem of the response of this mode to an external perturbation.

PROBLEMS

1. Try to perform analysis of the stability of the $M^{(1)}$ mode.
2. Analyse the period-doubling transition in terms of Fourier components (harmonics) of the height-time plot of the ball motion.

EXPERIMENT 6 *Mode $M^{(1)}$ doubles its period once more.*

1. Using the procedure described in Exp.6. put the ball into $\mathbf{M^{(1,2)}}$ mode.
2. Slowly increasing A find the threshold $A^{(1,2)}_{MAX}$ above which the $M^{(1,2)}$ mode gives place to its period-doubled version $M^{(1,2,2)}$.
3. Analyse the Poincare map portrait of the mode indicating which of its points represent which collisions within the height-time plots.
3. Try to see, what happens next. Are you able to observe the $M^{(1,2,2,2)}$ mode?

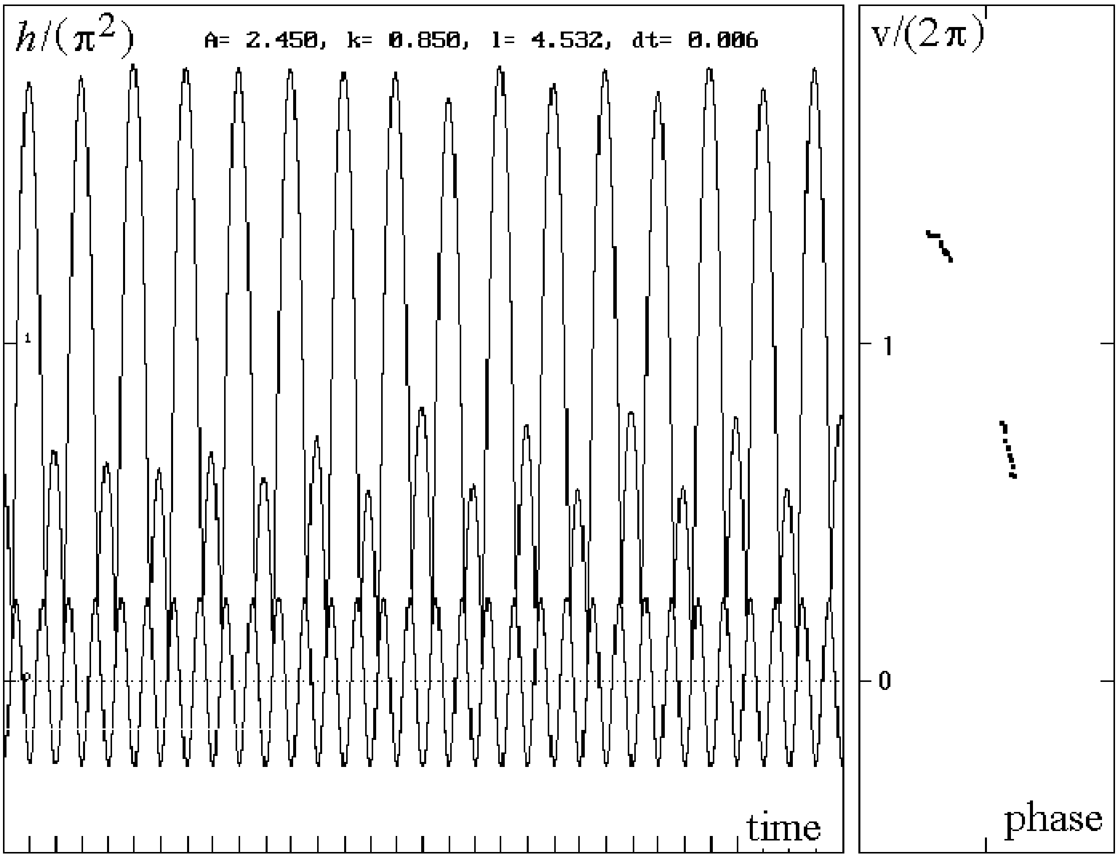

Fig. 17 As A goes above a certain threshold the simple period-doubled mode $M^{(1,2)}$ looses stability and doubles its period once more.

Period-doubling bifurcations described above can be easily reproduced in a laboratory experiment where a steel sphere is put into a sustained motion on the surface of glass or plastic plate attached to the membrane of a loudspeaker. When the frequency of the surface vibration is about 100 Hz, the sequence of collisions produces a clear sound. The sound changes in a distinct manner when the system goes through a period-doubling bifurcation: a lower (subharmonic) component is heard to appear.

As seen in the Fig., within the $M^{(1,\,2,\,2)}$ mode there are four types of collisions. Identify them within the Poincare map.

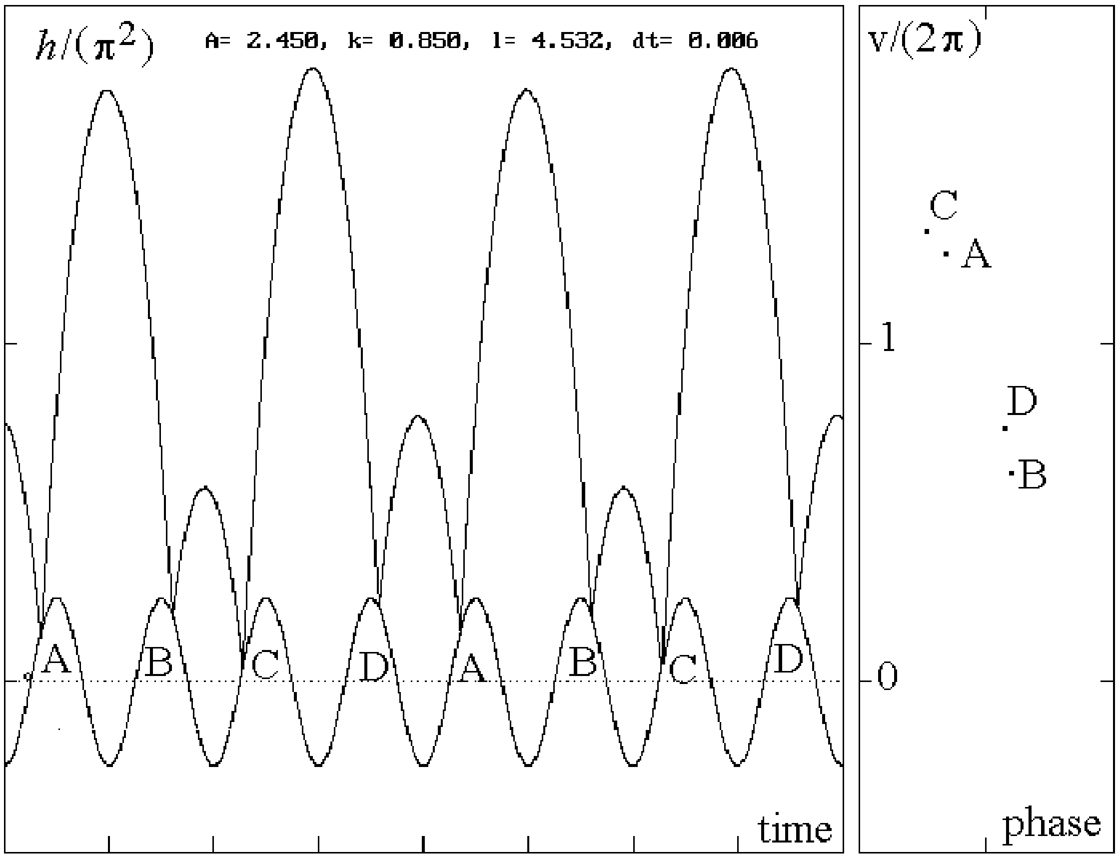

Fig. 18 Details of the ball motion within the $M^{(1,2,2)}$ mode.

EXPERIMENT 7 *Limits of stability of the $M^{(2)}$ mode*

1. Drive the ball into the $M^{(2)}$ mode.
2. Study its lower and upper stability limits: $A^{(2)}_{MIN}$ and $A^{(2)}_{MAX}$.

Fig. presented below shows the $M^{(2)}$ mode just above its lower stability threshold $A^{(2)}_{MIN}$. Note, that in search of the appropriate velocity of the vibrating surface (necessary to compensate loses of energy) the mode located its collision points almost on the maximum (positive) slope of the surface displacement function. When, at still decreasing A, the maximum slope point is reached, the mode looses its stability and ceases to exist. (See once more Exp.4.) Below $A^{(2)}_{MIN}$ the appropriate slope point cannot be found any more.

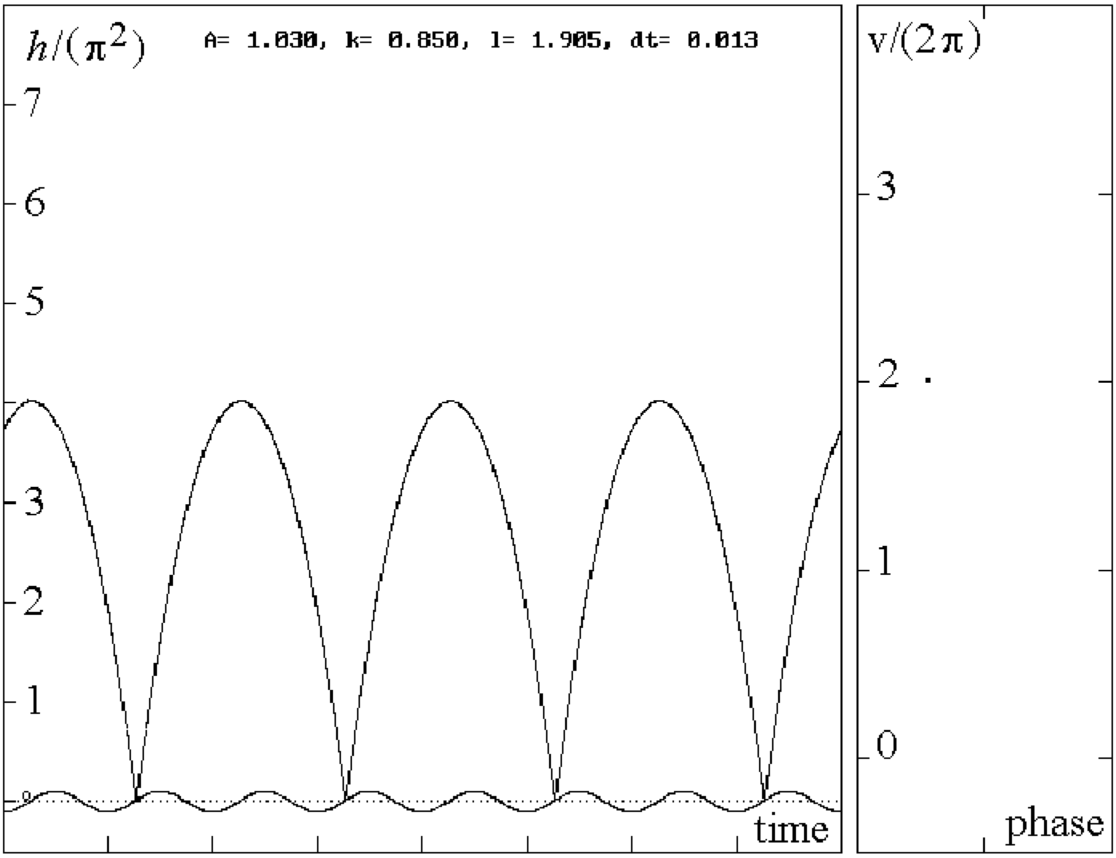

Fig. 19 $M^{(2)}$ mode just above $A^{(2)}_{MIN}$.

The scenario is different at the other end of the stability interval. When A increases, the $M^{(2)}$ mode shifts its collision points towards the maximum of the surface displacement function.

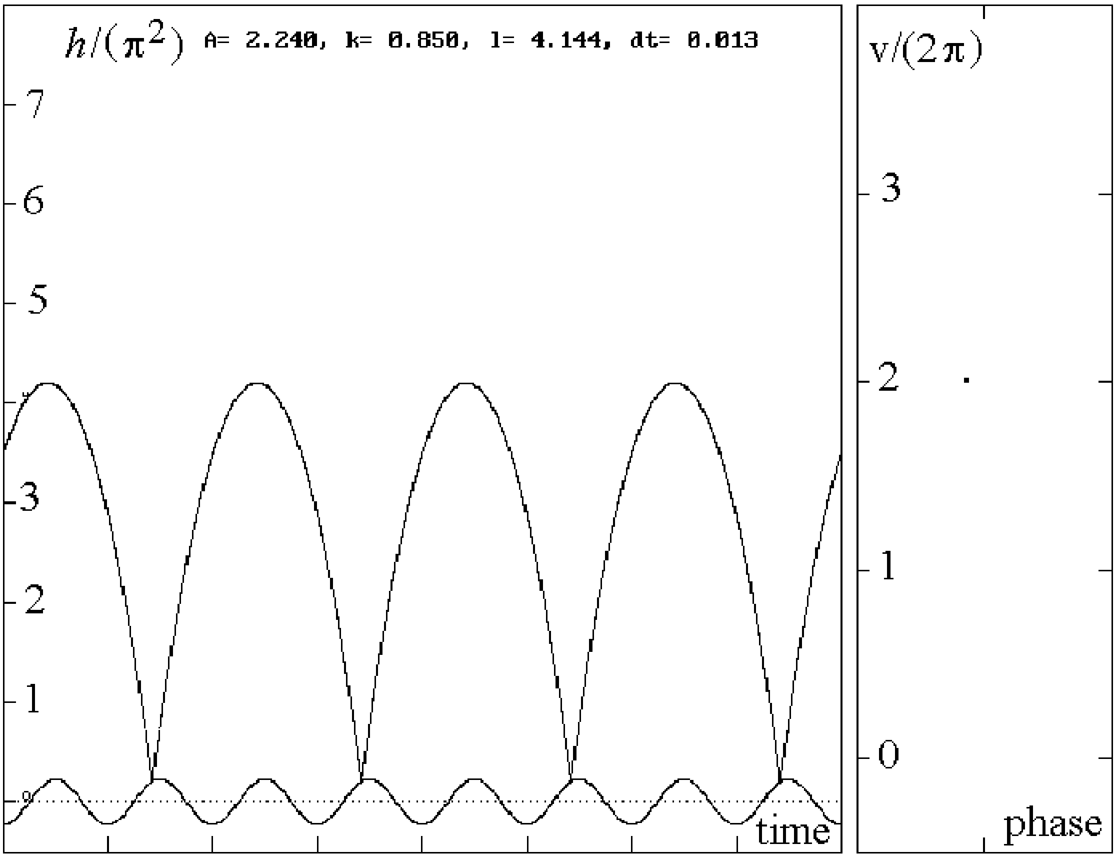

Fig. 20 $M^{(2)}$ mode just below the $A^{(2)}{}_{\mathrm{MAX}}$ threshold

This is necessary, since at increasing amplitude of the displacement the point with the appropriate slope (i.e. surface velocity) will be found closer and closer to the upper turning point. There is no limit to this kind of reasoning and we are forced to conclude that, no matter how large A, the $M^{(2)}$ mode will never cease to exist since the appropriate slope point can always be found. As we have already discussed it in Exp.5, the problem is that, although still existing, the mode can loose its resistance to perturbations and this is what happens at $A^{(2)}{}_{\mathrm{MAX}}$. Above the threshold, the $M^{(2)}$ mode is no longer stable. The story does not change in such a gloomy manner since at the moment, at which $M^{(2)}$ mode looses its stability, another mode is born. As in the case of the $M^{(1)}$ mode, the mode which is born at $A^{(2)}{}_{\mathrm{MAX}}$, is a period-doubled version of the destabilized one: $M^{(2,2)}$.

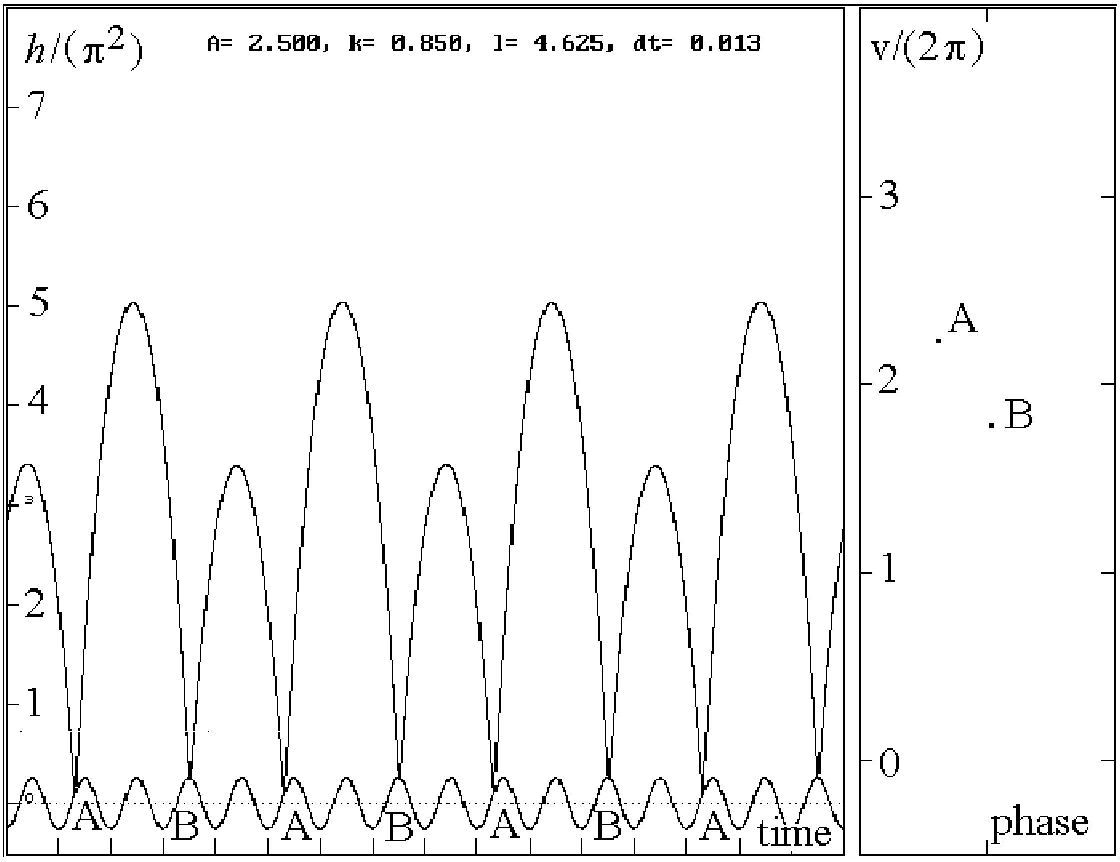

Fig. 21 Portrait of the $M^{(2,2)}$ mode: the period-doubled version of the $M^{(2)}$ mode.

PROBLEM

Find a general formula which describes the dependence of $A^{(n)}{}_{MIN}$ threshold versus k, where $n = 1,2,3, ...$ Is there any limit n_{MAX} for the order of the primitive period-n modes which can be sustained at a certain value of k?

Remark.

Experiments performed so far prove that for any periodic mode $M^{(n)}$ there exists an upper threshold of stability. Above such a threshold the $M^{(n)}$ mode evolves into its period-doubled version $M^{(n,2)}$. These period-doubling bifurcation thresholds are arranged in convergent sequence above whose limit $A_{MAX}(n, 2, 2, ...)$ all modes from a given sequence become unstable. Obviously, as one such cascade ends, the ball have a chance to find within the space phase another one, still stable. What we observe in such a case is that after an apparently chaotic transient it settles within the new cascade. And so on.

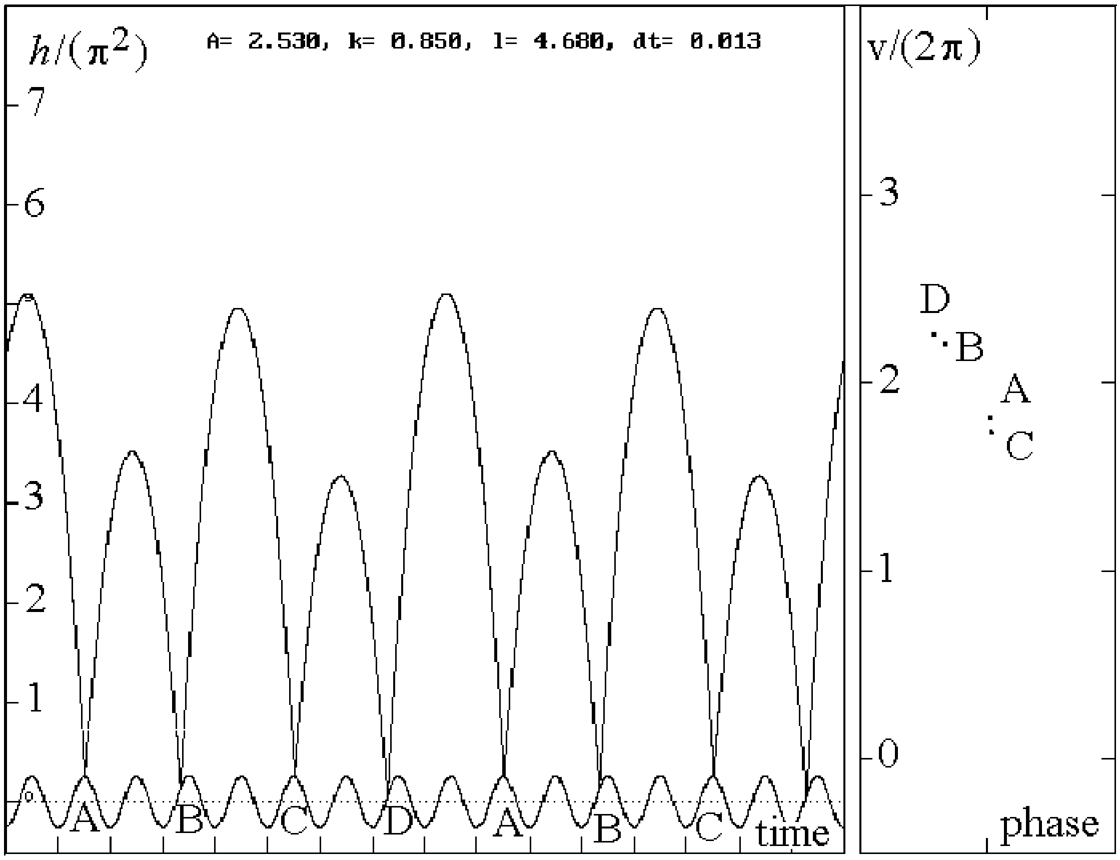

Fig. 22 It is possible to lead the $M^{(2,2)}$ mode above the stability limit where it gives place to its period doubled sister $M^{(2,2,2)}$.

EXPERIMENT 8 *The Feigenbaum ratio.*

1. Set k=0.85.
2. Drive the ball into the $M^{(1,\,2,\,2,\,2)}$ mode.
3. Slowly decreasing A find thresholds for three consecutive period-doublings: $A_{MAX}^{(1,2,2)}, A_{MAX}^{(1,2)}, A_{MAX}^{(1)}$.
4. Calculate the ratio:

$$d = \frac{A_{MAX}^{(1,2)} - A_{MAX}^{(1)}}{A_{MAX}^{(1,2,2)} - A_{MAX}^{(1,2)}}$$

Driving the ball into the delicate $M^{(1,2,2,2)}$ mode is not an easy task. The best (and almost single) way to achieve the task is to drive the ball into the $M^{(1)}$ mode and then, slowly increasing A, arrive to the $M^{(1,2,2,2)}$ mode via three consecutive period doubling bifurcations.

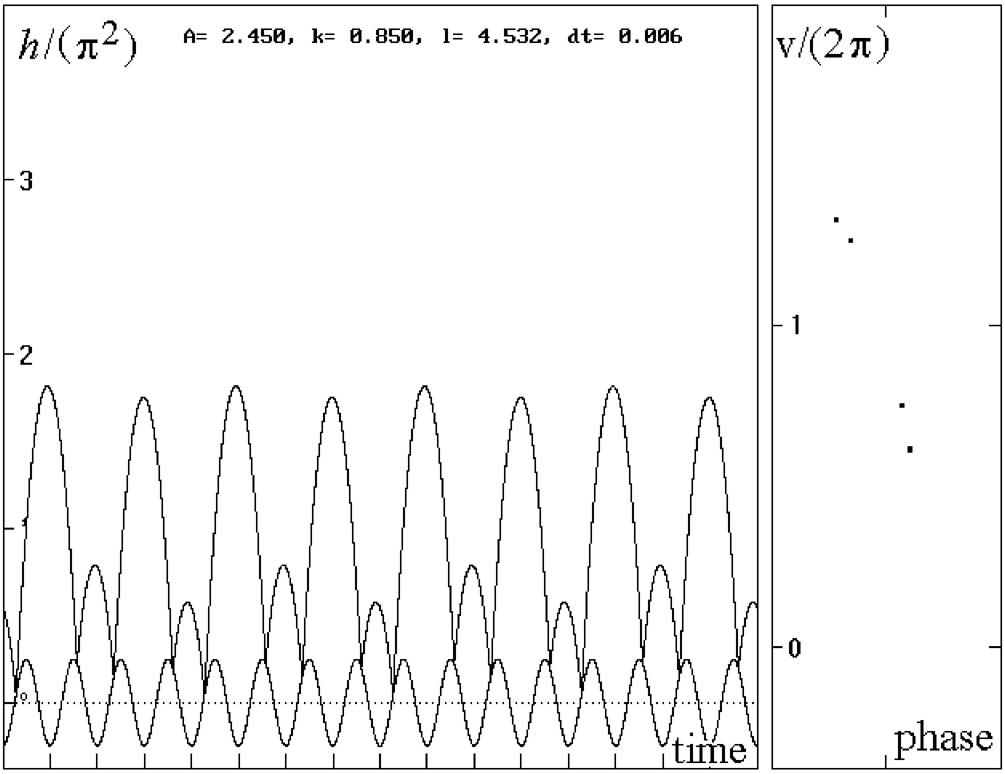

Fig. 23 $M^{(1,2,2)}$ mode just below its upper stability limit.

We do not advise, however, to determine localisation of the period-doubling bifurcation points on this route up A. The problem is that in the vicinity of the bifurcation points the dynamics of the relaxation processes, via which at any value of A the ball reaches its steady state of motion, becomes dramatically slow. As a rule, any of the periodic modes gets easily "overheated" i.e. while increasing A one enters the region where the mode is already unstable without noticing it. Then, all of a sudden one observes an abrupt transition to the new, stable mode, already well above its birthplace. As shown in: P.Pierański, M.Małecki, Nuovo Cim., 9D, 757 (1987), the overheating phenomenon is, at finite rates of the sweep of A, not only unavoidable, but, what is worse, unpredictable. The "overcooling" phenomenon, which occurs as A is decreased, although also unavoidable, is completely predictable: it can be seen simply as a time lag.

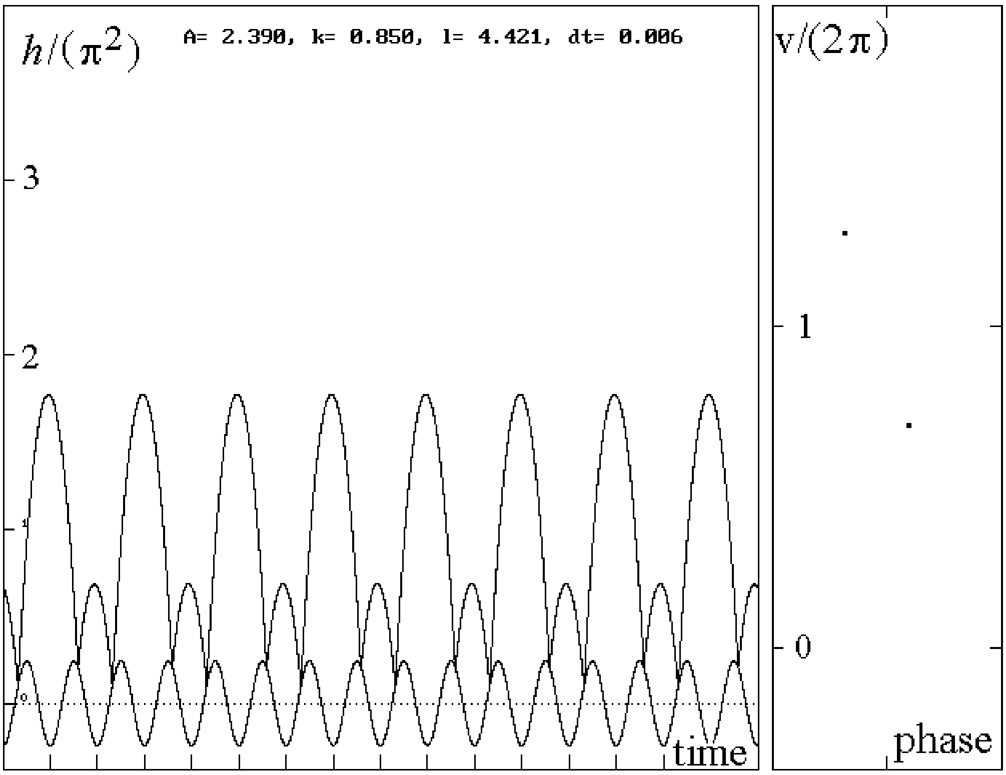

Fig. 24 $M^{(1,2)}$ mode just below its upper stability limit.

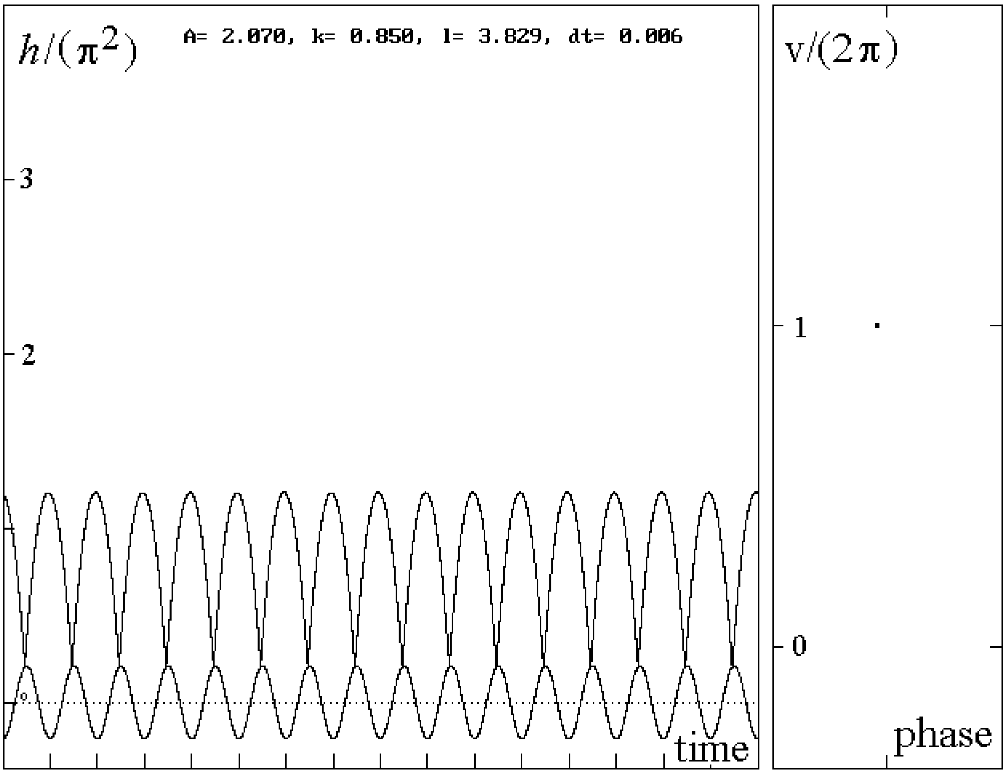

Fig. 25 $M^{(1)}$ mode just below its upper stability limit.

Fig.s presented above show trajectories of the ball just below the three period-doubling thresholds. As easy to see, the values of *A* at which the bifurcations take place are arranged into a convergent sequence. A precise, quantitative study proves that the sequence is convergent geometrically and the limit ratio δ_∞ at which distances between its consecutive points shrink is universal for the whole class of the dissipative systems. Since the ratio itself and its universality have been discovered by Mitchel Feigenbaum[1], it is often referred to as the "Feigenbaum ratio": d^{DIS}›Ą = 4.669...

The experiment you performed gives its rough estimate.

1 M.Feigenbaum, J.Stat.Phys. <u>19</u>, 25 (1978)

<u>EXPERIMENT</u> 9 *Can modes of motion peacefully coexist* ?

1. Set k=0.85 and A=1.87.
2. Drop the ball several times from height 1 (key 1) and identify its modes of motion. Try many times.
3. Repeat the experiment dropping the ball from different heights. Try to detect all possible modes of motion the ball is able to enter.

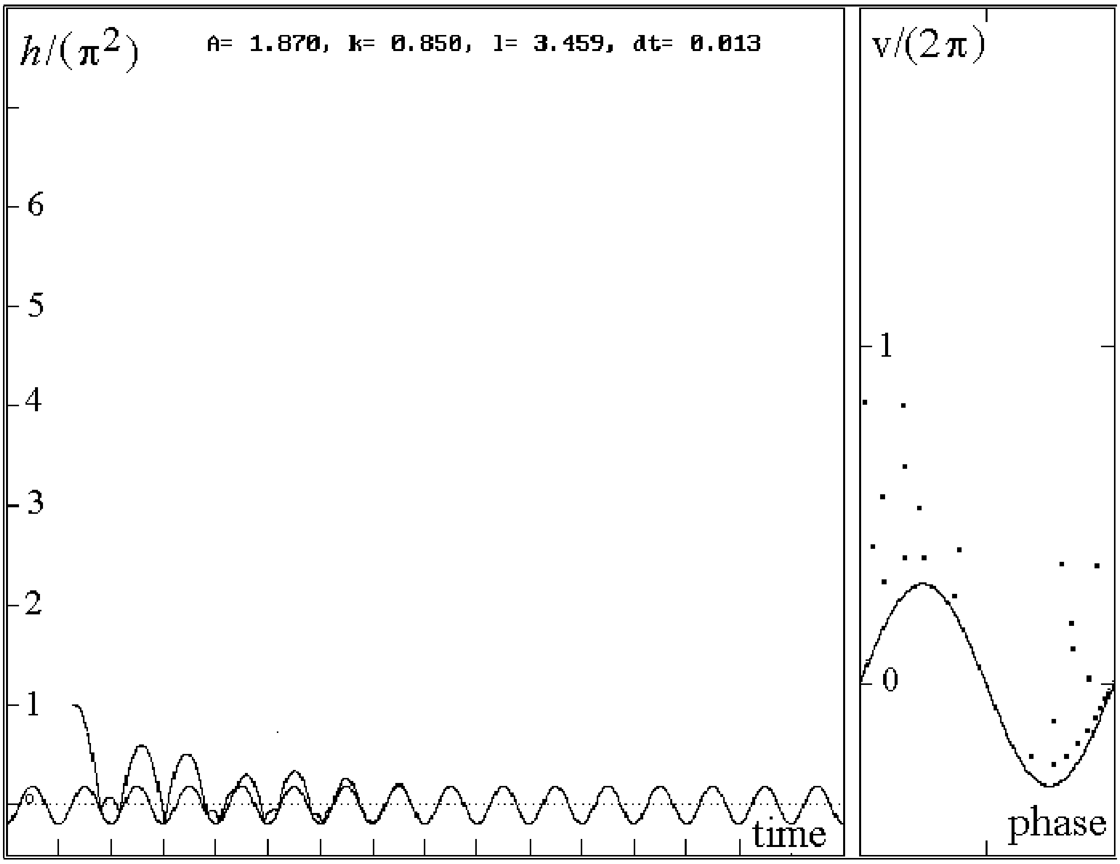

Fig. 26 A transient via which the ball dropped from height 1 enters the mute $M^{(0)}$ mode.

As seen in the Fig. above, one of the few possible results of the experiment in which the ball has been dropped from height h=1 is ... a failure to sustain its motion. After an initial transient the ball enters a convergent sequence of bounces, comes to the permanent contact with the moving surface and remains in it forever: the mute $M^{(0)}$ mode has been entered.

Another possibility is illustrated in the Fig. below. It may happen that the motion of the ball will be sustained and via an initial transient it will enter its simplest, periodic $M^{(1)}$ mode we have studied in previous experiments.

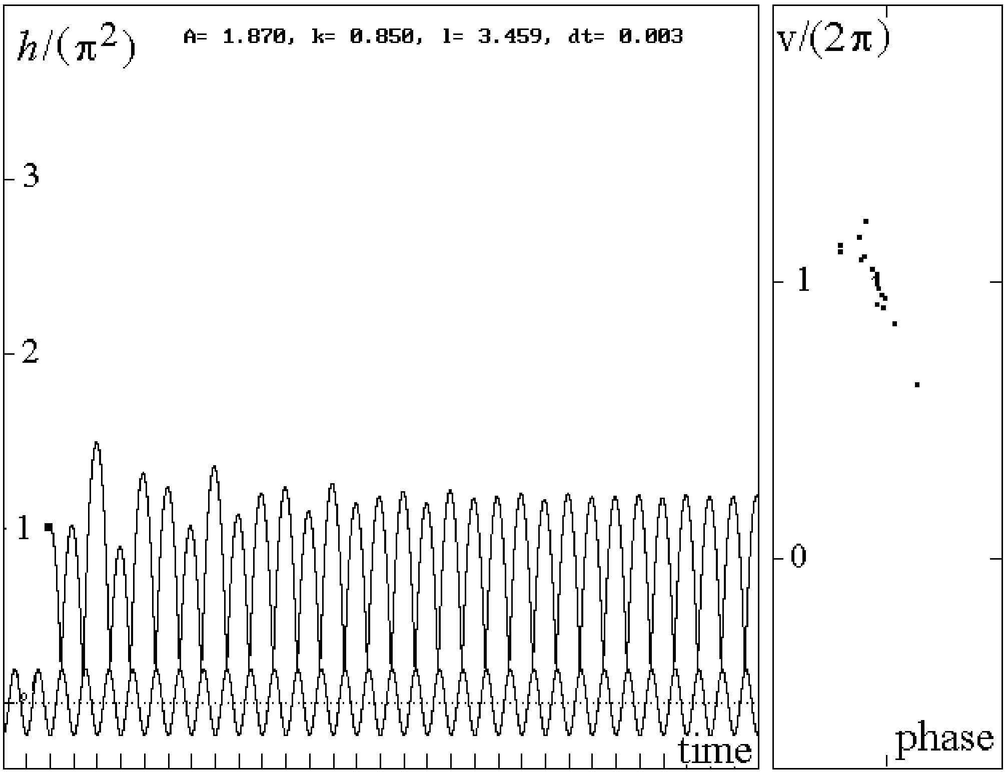

Fig. 27 A transient via which the ball dropped height 1 enters its simplest $M^{(1)}$ periodic mode.

Sometimes, however, the ball behaves still in a different manner. From the very beginning one can see that it will neither sit down on the surface nor it will enter the $M^{(1)}$ mode. What happens instead is that it enters a periodic sequence consisting of equal triples of jumps: a small jump, a larger one and still a larger one. Altogether, the jumps last three periods of the surface vibration. Within the Poincare map the mode is represented by a cycle of three points surrounding the single point which represents the $M^{(1)}$ mode. From a certain point of view this mode of motion can be seen as a modulation of the $M^{(1)}$ mode: we denote it as $M^{(1,3)}$ mode. In the next experiments we shall study its properties.

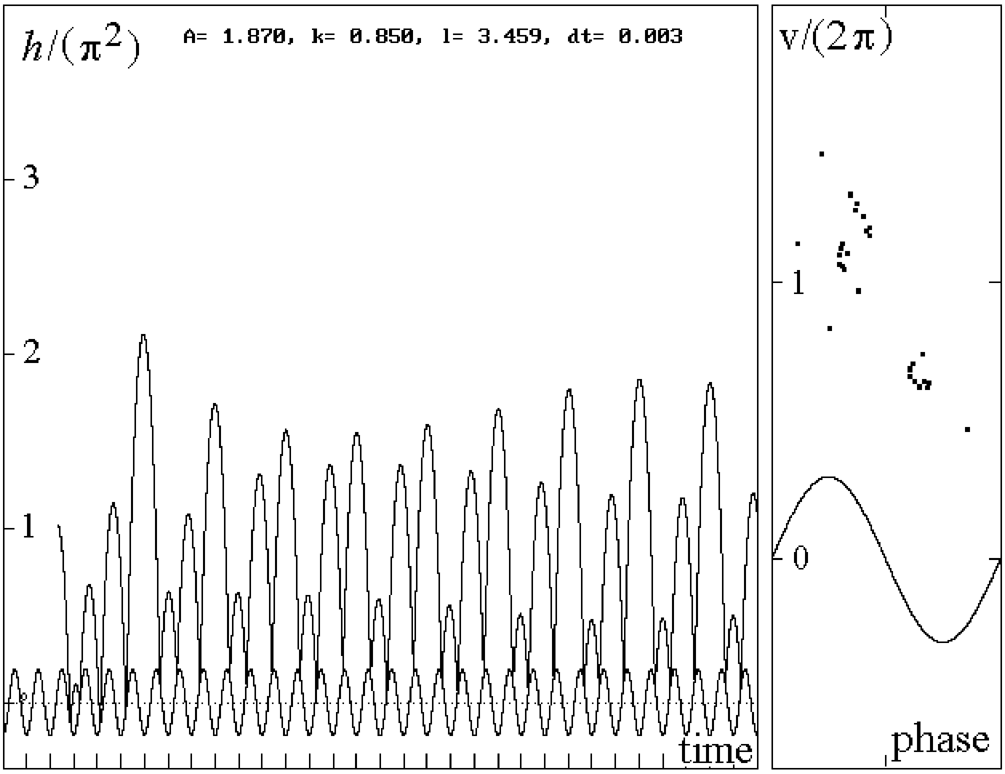

Fig. 28 Sometimes, however, when dropped at an opportune phase of the surface vibration, it chooses to settle in a different mode of motion - $M^{(1,3)}$.

If without changing the amplitude of the surface vibration one starts dropping the ball from larger heights, one can still find new modes of motion. Next Fig.s show two of them. The first one, shown in Fig.20, is very similar to the $M^{(1)}$ mode. It also consists of equal jumps; this time , however, each of them is four times as high and two times as long as a single jump from the $M^{(1)}$ mode; the ball meets the collision surface every second period. That is why we denote it $M^{(2)}$. Within the Poincare section the mode is represented by a single point located at v=2π, twice as high as the point which represents the $M^{(1)}$ mode. There is no problem to find this mode in a real experiment. It can be easily recognised since not only the sound it produces is by an octave lower in its pitch but (and this is quite spectacular) the ball is seen to jump much higher. At a frequency of the surface vibration about f_S=10 Hz the jumps are quite high and can be easily analysed with a stroboscopic lamp blinking with a frequency slightly different from f_S.

PROBLEM

1. Calculate the height of a single jump of the ball in the real experiment, where f_S=50 Hz and g=9.81 [m/s^2].

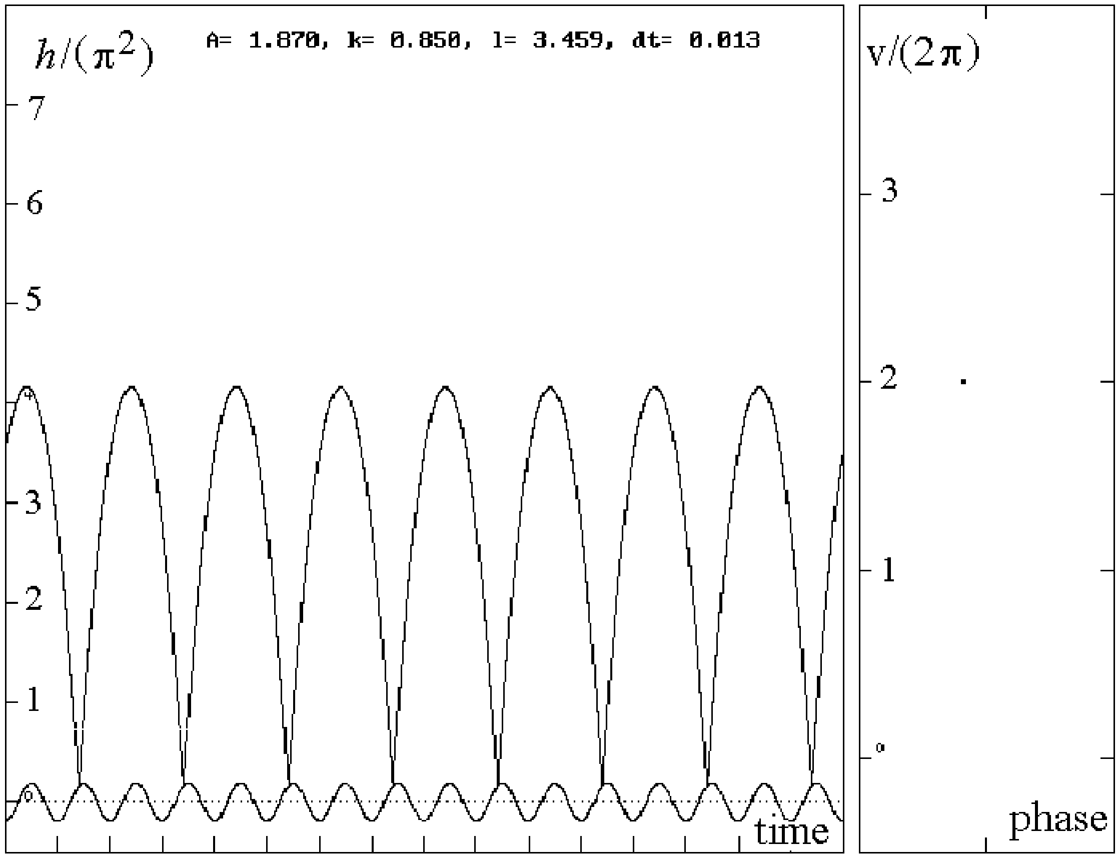

Fig. 29 Portrait of the $M^{(2)}$ mode.

Repeating many times the experiment in which the ball is dropped from height 4 one can observe a variation of the $M^{(2)}$ mode analogous to the $M^{(1,3)}$ mode. Fig. below presents its portrait. We leave its analysis to the reader.

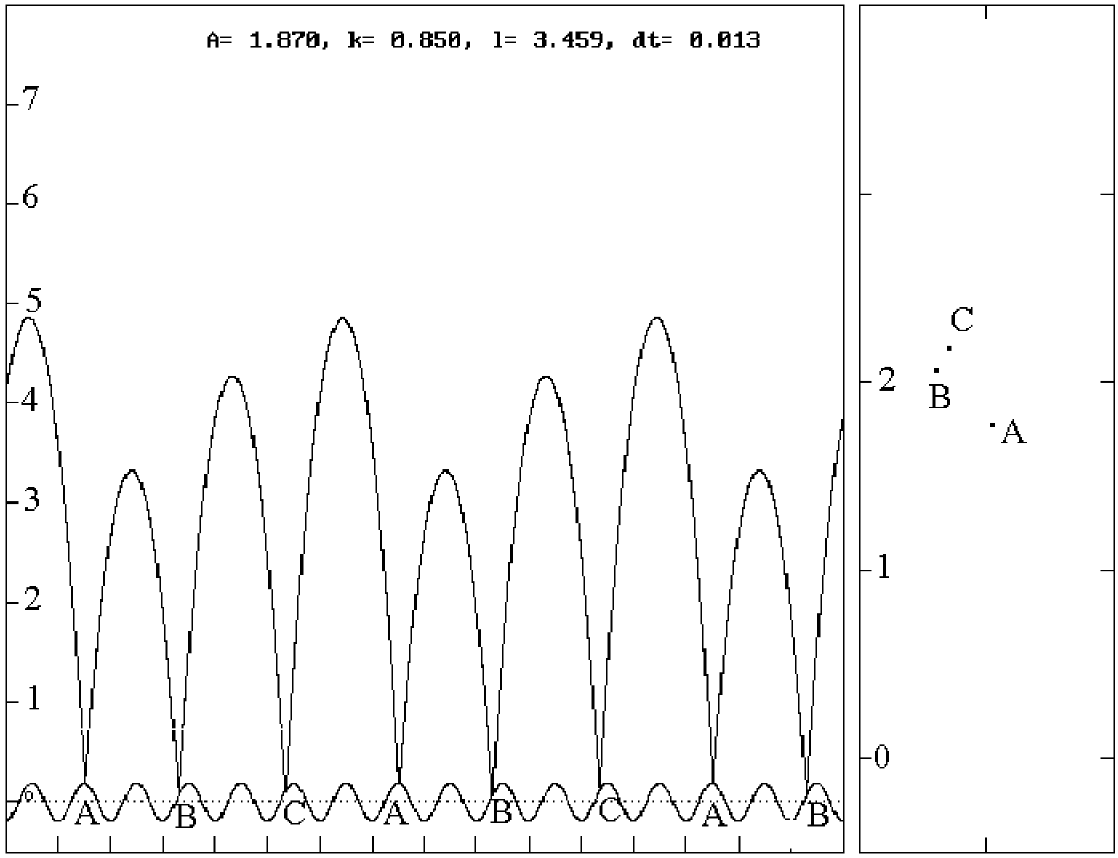

Fig. 30 When trying hard, one is able to drive the ball into still another mode. As easy to see, it is a period-6 modification of the $M^{(2)}$ mode.

As the experiment indicates, in contrast to linear systems, the bouncing ball can for the same values of its control parameters enter a few, distinctly different modes of motion. Within the Poincare map the modes are seen as coexisting attractors: each with its own basin of attraction. Dropping the ball from different heights, at different phases of the surface motion one is able to enter basins of different modes.

From a particular point of view, the finite family of steady modes observed for the dissipative BB model can be seen as a remainder of the infinite hierarchy of rational invariant tori found within the phase space of the system in its conservative, k=1, version.

EXPERIMENT 10 *Mode $M^{(1,3)}$ doubles its period.*

1. Using procedure described in Exp.7 drive the ball into its $M^{(1,3)}$ mode.
2. Increase slowly A and observe evolution of the motion image within the Poincare .

The aim of this experiment is to find out if the how the $M^{(1,3)}$ mode behaves as the amplitude of the surfaces vibration increases. Previous experiments have taught us that a simple periodic mode $M^{(n)}$ can loose its stability and in a smooth manner give place to its period doubled variation $M^{(n,2)}$. Will anything of this sort happen to the triple-periodic $M^{(1,3)}$ mode? Fig. below presents portrait of the mode just below the point where it is to loose it stability.

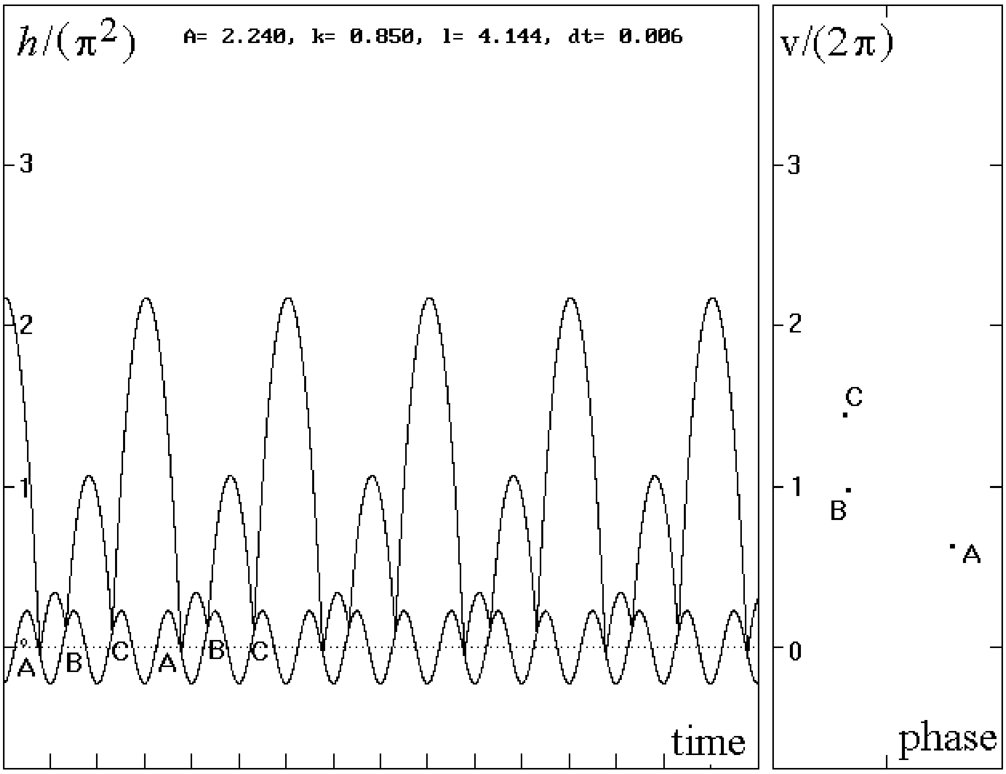

Fig. 31 The $M^{(1,3)}$ mode just below its period doubling bifurcation.

Next Fig. presents what happens above the threshold. As easy to see, also $M^{(1,3)}$ mode is able to double it period. Since, however, it period was already $3T$, after the bifurcation it becomes equal $6T$. (T is the period of the surface vibration.) Now, the sequence of jumps consists of "words" each of which consists of two slightly different triples. We denote the mode by $M^{(1,3,2}$

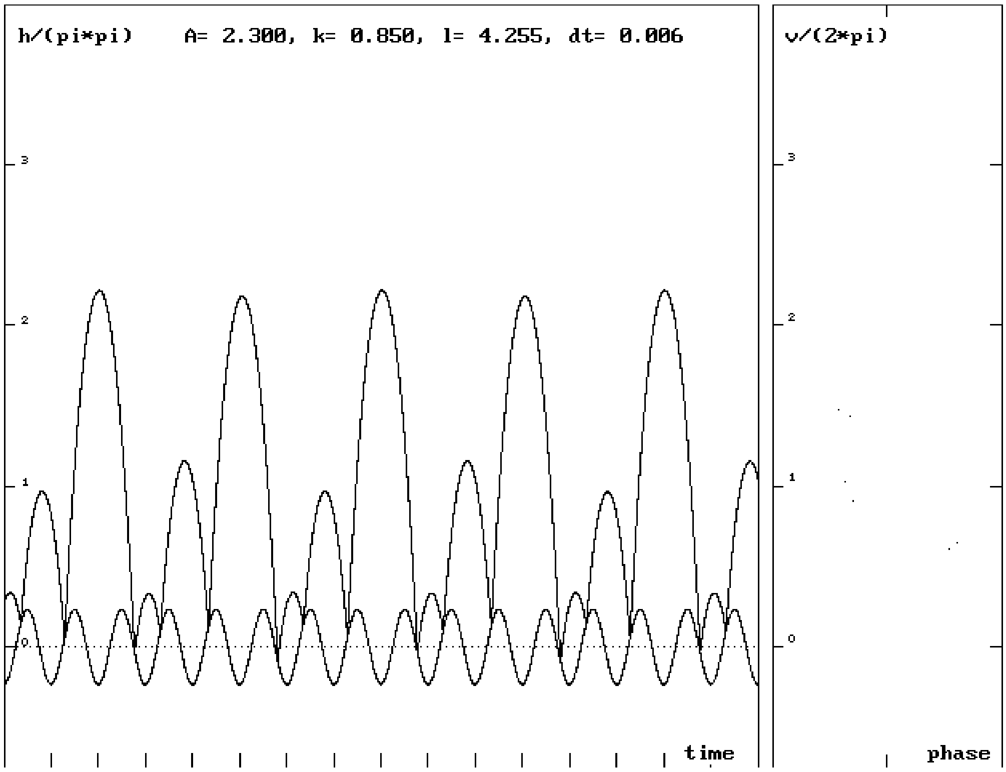

Fig. 32 Above a well defined threshold $A^{(1,3)}_{MAX}$ the $M^{(1,3)}$ mode looses stability and bifurcates into its period-doubled version $M^{(1,3,2)}$.

PROBLEMS

1. Find the relation between collisions seen in the height-time plot and the points which represent them within the Poincare map.
2. Are you able to observe next period-doubling bifurcation of the mode?

<u>EXPERIMENT</u> 11 *Where and how the* $M^{(1,3)}$ *is born* ?

1. Using procedure described in Exp.8 drive the ball into its $M^{(1,3)}$ mode.
2. Decreasing slowly A observe its evolution.

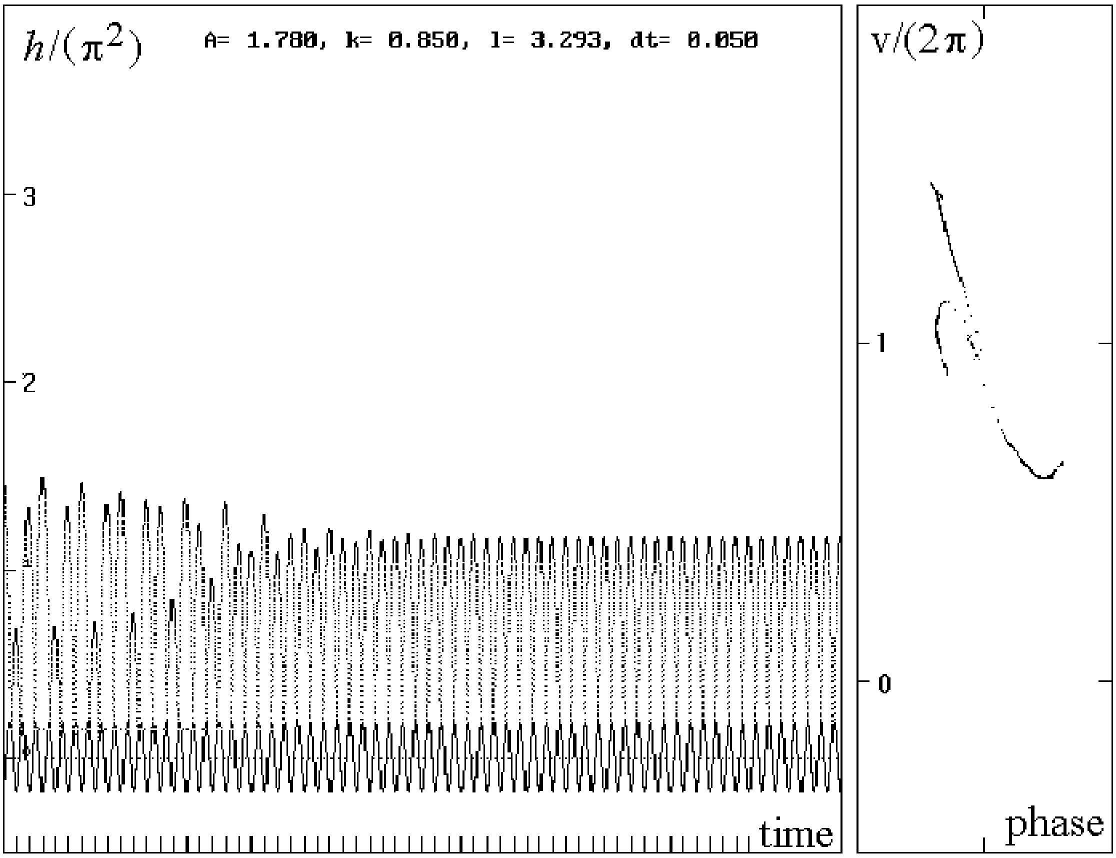

Fig. 33 When A decreases, the $M^{(1,3,2)}$ mode is seen to turn back into $M^{(1,3)}$ mode which then, in an abrupt manner, evolves into the $M^{(1)}$ mode.

In contrast to the $M^{(1,2)}$ period-doubled mode, the period-3 $M^{(1,3)}$ mode does not evolve in a continuous manner from $M^{(1)}$ mode. As shown in the Fig. above, the $M^{(1,3)}$ mode looses its stability still at a certain distance from the $M^{(1)}$ mode. Consequently, the $M^{(1)}$ - $M^{(1,3)}$ transition should be considered as of the "first order" while that $M^{(1)}$ - $M^{(1,2)}$ of the "second order".

EXPERIMENT 12 *Chaos. Does it remember order*?

1. Keeping k constant increase A to such a value at which none of the periodic mode is remains stable.
2. Identify within the chaotic trajectory intervals of time during which the trajectory becomes similar to one of the previously studied periodic mode (in spite of their being unstable).

As mentioned at the end of the description of Exp.6 although becoming unstable above its period-doubling bifurcation point, any periodic mode does not vanish from the phase space of the system. It remains there as a cycle of (unstable) fixed points repelling trajectories passing nearby.

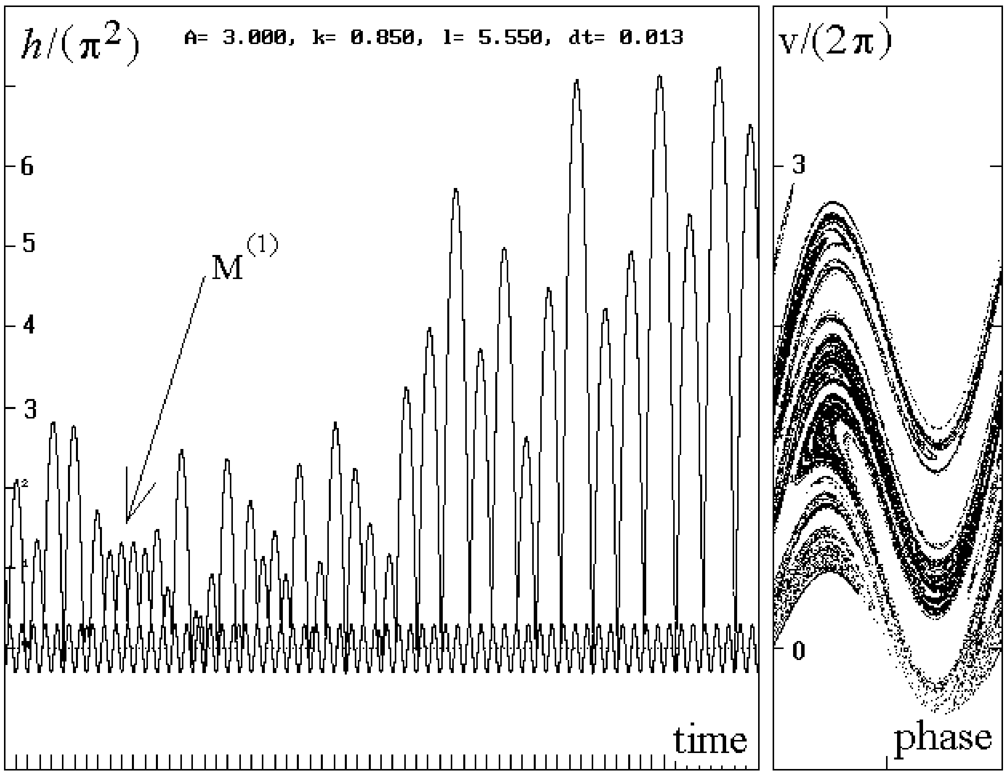

Fig. 34 A part of the chaotic trajectory. Still, however, one can distinguish within it a short subsequence of jumps reminding the $M^{(1)}$ mode.

When a chaotic trajectory passes nearby such a cycle, it becomes for a while similar to the periodic trajectory which the cycle represents. Since for large values of A there are within the phase space quite a few such unstable cycles (an infinite number, to be precise) one observes quite often within the chaotic trajectory a shorter or longer almost periodic transient. Fig.s presented above and below show two examples of such intervals.

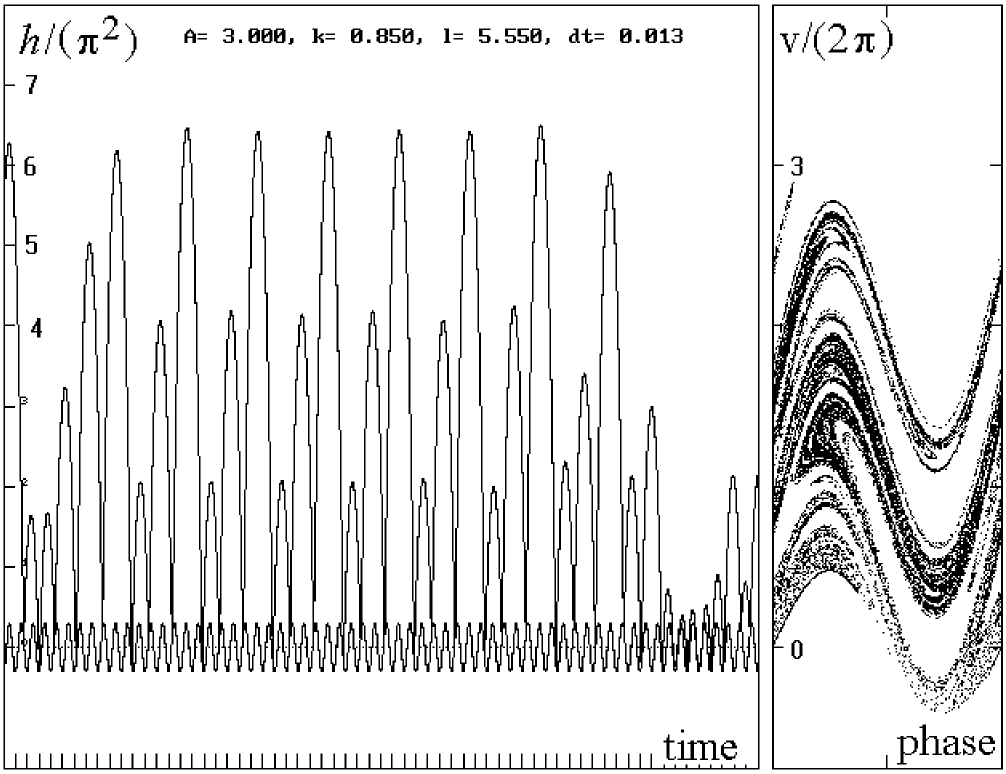

Fig. 35 Another part of the same chaotic trajectory. This time the trajectory passes nearby the unstable $M^{(2,3)}$ mode.

Chaotic trajectories appearing within a deterministic system do not fill in a random manner its phase space. As seen in the Poincare map,, they fall onto a complex, but well defined and bounded object. Due to its unusual properties, of which fractal dimensionality is the most well known, it is called "strange attractor". Note that in the particular case we study here the attractor is sharply cut at its bottom by a sine-shaped boundary.

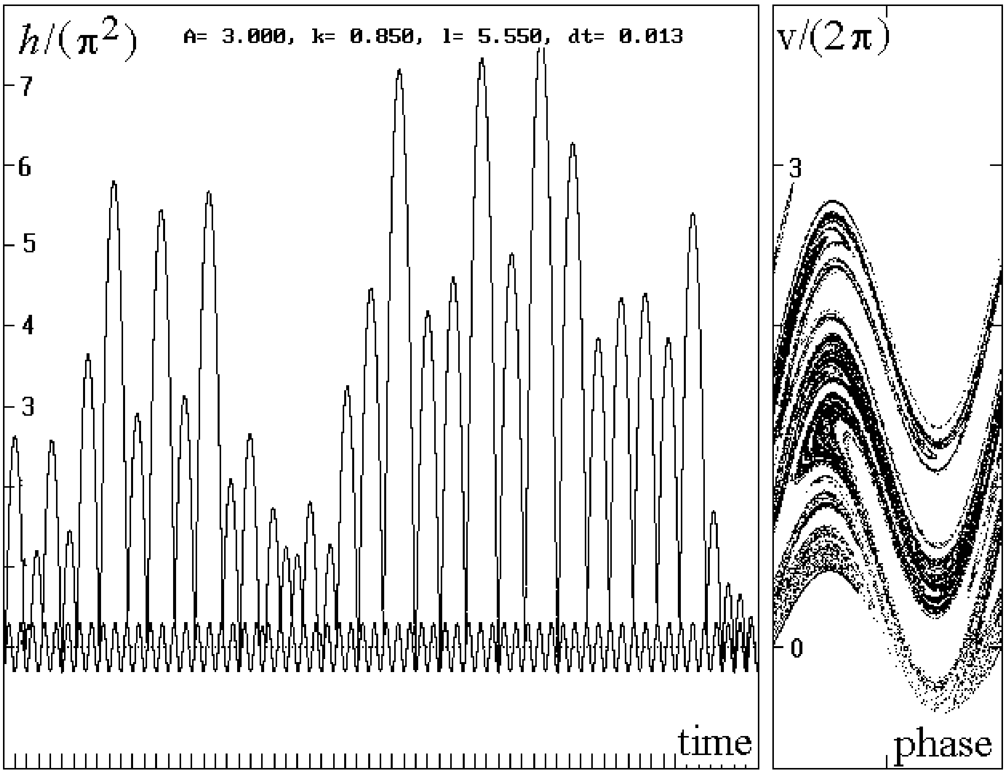

Fig. 36 Still another part of the same trajectory. Does it contain any piece similar to one of the periodic modes we have already studied?

PROBLEMS

1. Discuss the physical meaning of the sine-shaped lower boundary. Find its equation.
2. Is there for given A and k any upper limit within the Poincare map above which no trajectory can arrive?

<u>EXPERIMENT</u> 13 *The chirping mode*.

1. Using procedure described in Exp.2. drive the ball into the mute $M^{(0)}$ mode.
2. Increasing slowly A find the threshold $A^{(0)}{}_{MAX}$ above which the mode looses stability and evolves into the chirping $M_1(0)$ mode.
3. Find the upper limit for the stability of the $M_1(0)$ mode.

Within the $M^{(0)}$ mode the ball moves together with the surface, sitting on it. When however the amplitude of the surface vibration becomes too large, the inertia force may exceed the gravitational one. In such a case, the ball will in a smooth manner leave the surface getting into a free flight which initiates a sequence of bounces converging to the surface still within the same period of the surface motion.

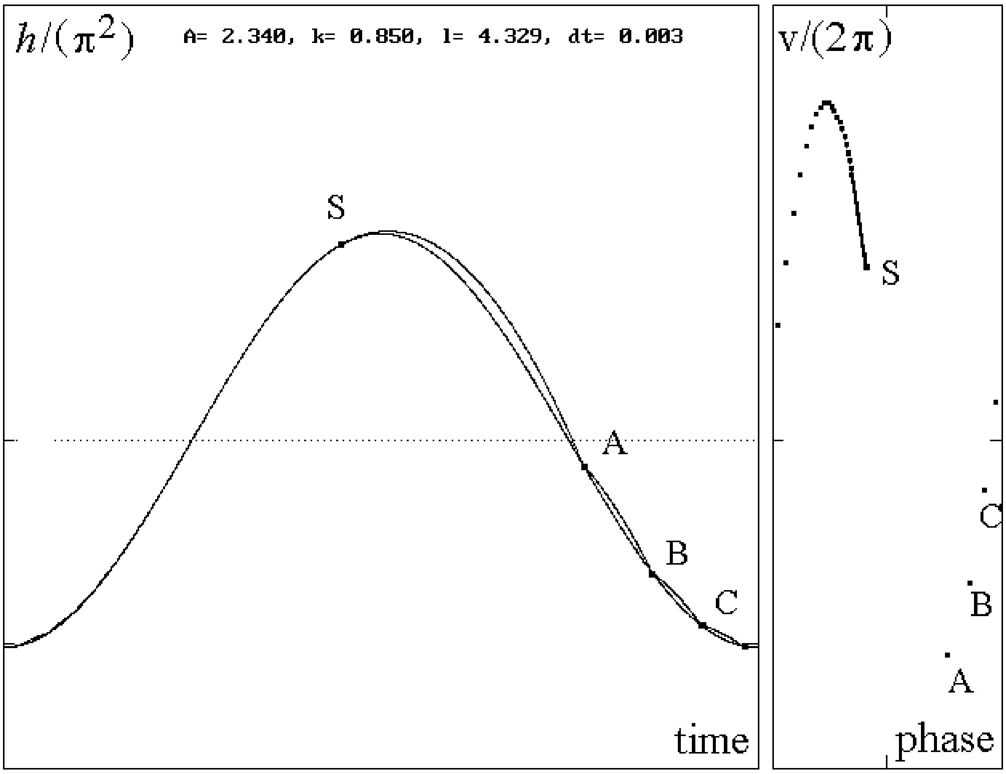

Fig. 37 A single period of the chirping mode $M^{(0)}{}_1$ at k=0.85. Note how close to the surface the ball is moving.

This repeats in any period of the surface vibration. Sound produced by this mode of motion reminds chirping of a grasshopper - that is why it has been called: the *chirping mode* . Fig. above presents its details.

Note, that in the k=0.85 case the trajectory of the ball passes very close to that of the vibrating surface, thus, its collisions with the surface occur at small relative velocities. As a result such a mode can be hardly audible in nature. For smaller values of k, i.e. for stronger dissipation, the jumps of the ball within the chirping mode become relatively higher. See Fig. below. Obviously, in such a case they produce a louder sound.

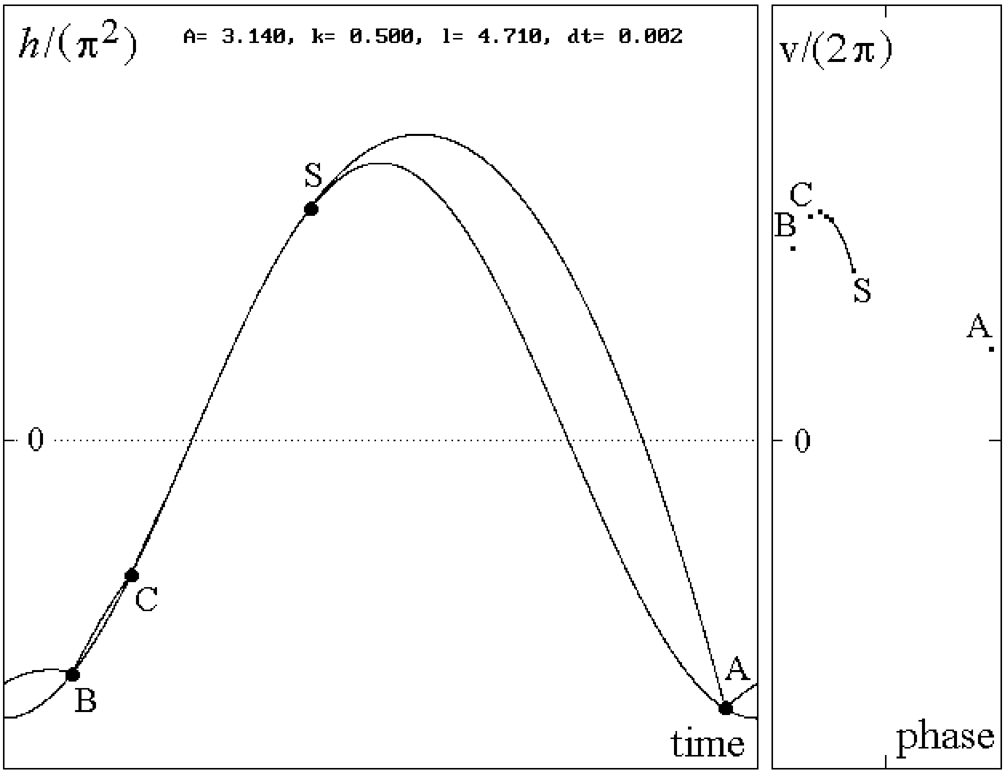

Fig. 38 The chirping mode for a smaller value of k. Although in comparison with Fig. 35 the first jump is higher, the sequence of jumps does not last longer.

Note also, that as A increases, the beginning and end of the sequence of bounces converge to each other (the former shifts backwards while the latter shifts forward). When they meet, at $A_{1MAX}(0)$, the $M^{(0)}{}_1$ mode looses stability.

PROBLEMS

1. Find the formula for the value of A at which the chirping mode is born.
2. Find out what happens to the chirping mode above its upper stability limit.
3. Is the existence of the chirping mode of importance in those regions of A at which chaotic trajectories are observed?

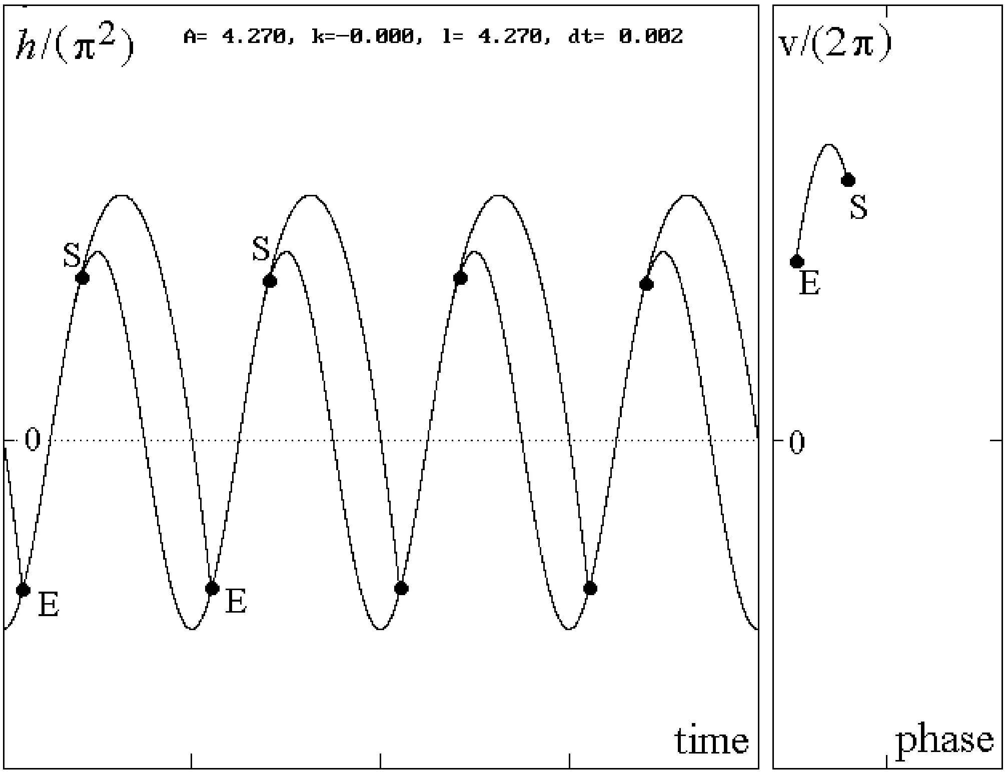

Fig. 39 The chirping mode at *k*=0. Acceleration of the surface makes the ball to leave it and enter a free flight. This time, however, there are no further bounces.

V. CONSERVATIVE BOUNCING BALL

It is impossible to design an experimental BB model in which *k* would be exactly equal to 1. On the other hand, the simulated BB workbench allows one to reach the limit just by pressing a single key. Observation of motions of the conservative BB allows one to grasp proper intuitive understanding of such notions of the Hamiltonian dynamics as *invariant rational and irrational tori, KAM theorem, elliptic and hyperbolic fixed points, stochastic layer* etc. A detailed study of these items needs a separate, extensive sequence of experiments. Here, we describe but a single experiment and suggest a few problems the reader can study on his/her own.

Structure of the phase space of a conservative system is essentially different from that of a dissipative one. In presence of a nonlinearity (in the BB model provided by the sine-shaped vibrations of the collisions surface) an infinitely complex hierarchy of elliptic and hyperbolic fixed points makes the skeleton of a complex structure within which stochastic layers and invariant irrational tori (those, which managed to survive) are neatly distributed. As the nonlinearity increases, more and more of the irrational tori are destroyed and the web of the stochastic layers becomes thicker (KAM theorem). The whole process is a complex as the structure itself but the experiment we suggest below allows one to grasp its physical meaning. We encourage the reader to experiment on his/her own.

Studying the structure of the phase space one needs getting into its different parts. Keys 1,2,... are here very useful. When a subtle change of the position is needed one should rather play with the value of *k*; changing for a while its value below or above 1, one is able to drive the trajectory in almost any corner of the Poincare map.

Integration of the equations of motion are in the conservative limit (where for large *A* the bounces of the ball may become very high) rather difficult. Since the integration algorithm we applied is deliberately simple it is necessary to pay attention to the value of *dt* - it should be never too big.

<u>EXPERIMENT</u> 14 *Search for the typical modes of motion of the conservative bouncing ball.*

1. Set k=1, A=0.1 .
2. Drop the ball from different heights and observe its trajectories.
3. Analyse traces of different trajectories within Poincare map.

Dropping the ball from different heights and/or switching for a while values of k different from 1 allow one to search the whole plane of the Poincare map. Fig. below presents results of such an experiment.

At the bottom of the plot one finds a family of trajectories which with a good accuracy follow at a certain distance the lower boundary of the plot - trace of the mute mode. One of such trajectories is shown on the height-time plot of the same Fig.. It consists of tiny jumps located on the cosine shaped plot of the surface motion.

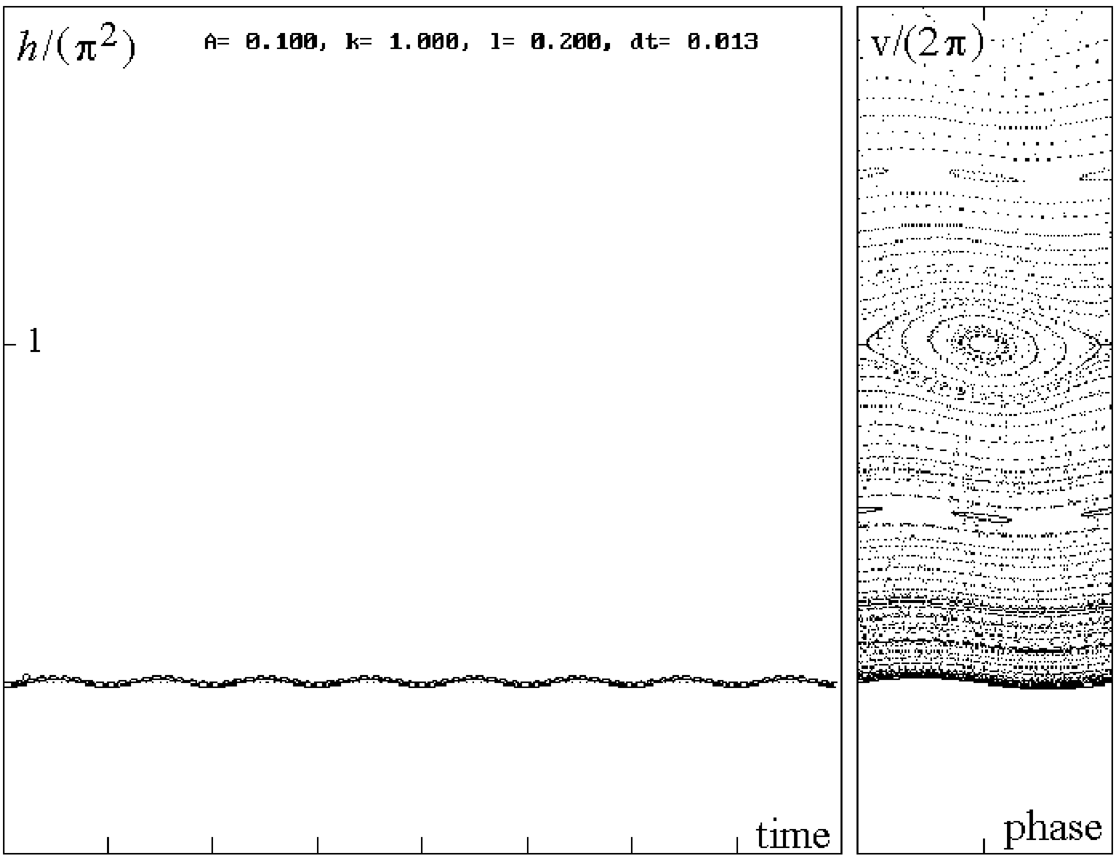

Fig. 40 Structure of the Poincare map in the Hamiltonian case. The trajectory shown within the height-time plot consists of almost equal, very small jumps.

Analysing the Fig., one can conclude that when the jumps are very low their response to the slowly (in terms of the period of a single jump) varying position of the surface is simply following it up and down. This is what theoreticians call an "adiabatic limit".

The solution is dramatically different when the jumps are of such a height that their time length becomes comparable to the period of the surface vibration. As one can find out, when the jumps are exactly of such a height that their time length is equal to the period T_S of the surface vibration and when collisions with the moving surface occur exactly either at the upper or at the lower turning points (where the velocity of the surface is equal to zero), then the ball does not even notice that the surface is moving. As a result, one observes sequences of perfectly equal jumps. Within the Poincare map the two solutions are represented by single points: one of them - *elliptic*, the other one - *hyperbolic*. Fig.s presented below show two trajectories passing near the simplest elliptic fixed point.

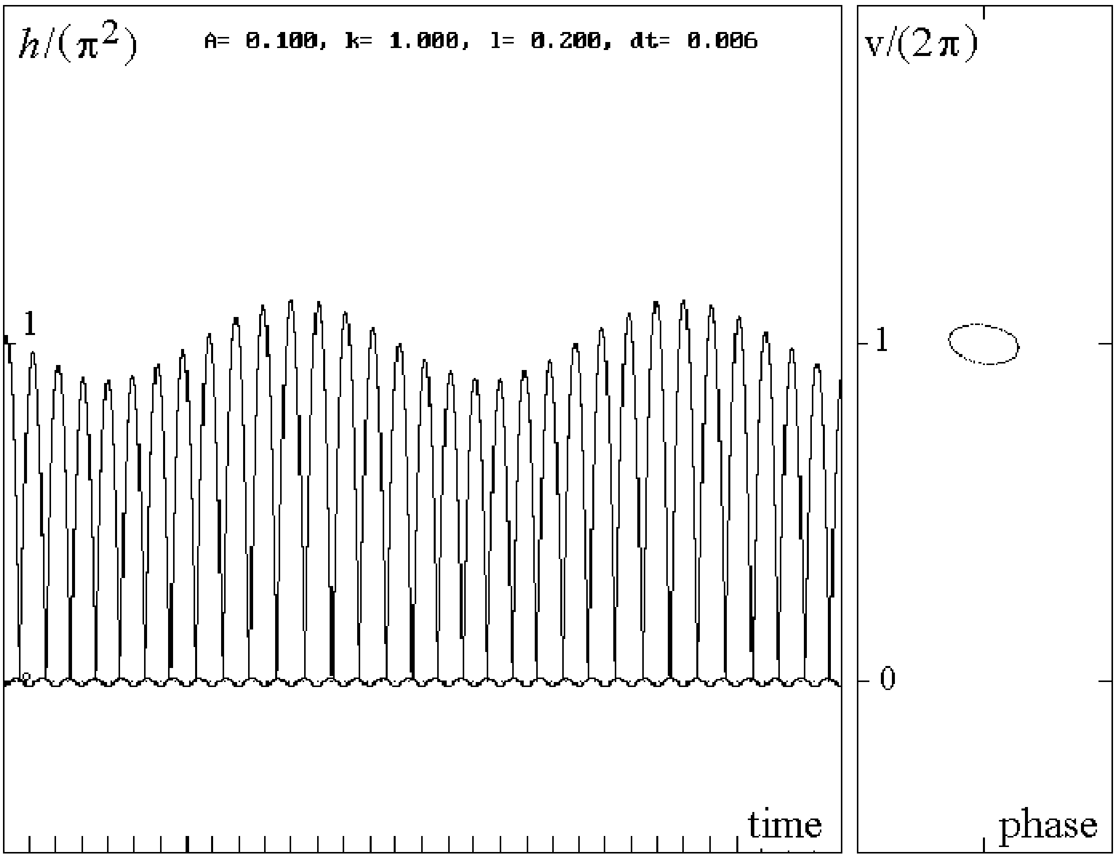

Fig. 41 A single trajectory near the elliptic fixed point.

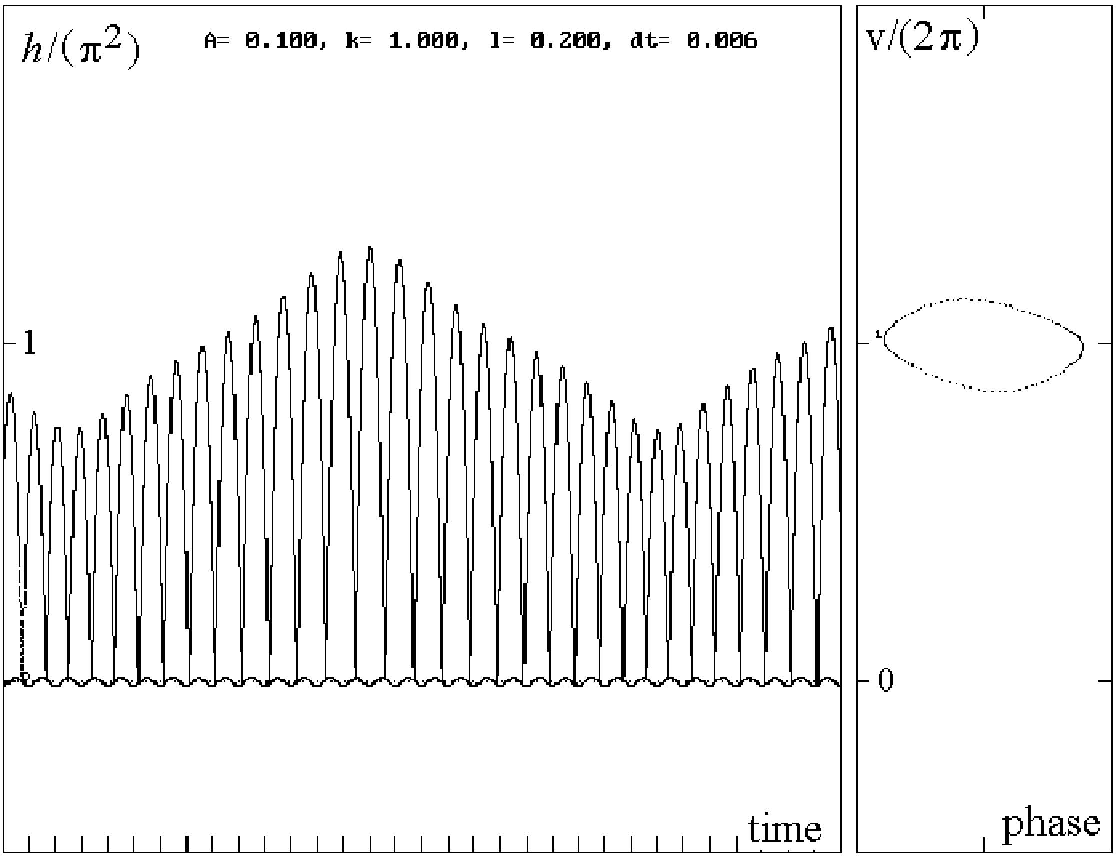

Fig. 42 Another trajectory surrounding the elliptic fixed point. This time, however, it passes nearby the hyperbolic fixed point.

Trajectories presented in the above Fig.s can be seen as phase modulated versions of the ideal, period-1 trajectory represented by the fixed point. Note, that the phase modulation within the trajectory located closer to the fixed point is sine-shaped, i.e. it should be well described by a linear approximation, while that passing closer to the hyperbolic fixed point is strongly non-harmonic.

As dissipation is switched on, the phase oscillations would become damped and the elliptic fixed point would turn into the point attractor representing within the phase space the $M^{(1)}$ mode.

Fig.s presented below show consecutive pieces of a single trajectory passing within the so-called *stochastic layer* within which the hyperbolic fixed point (0, 2π) is submerged.

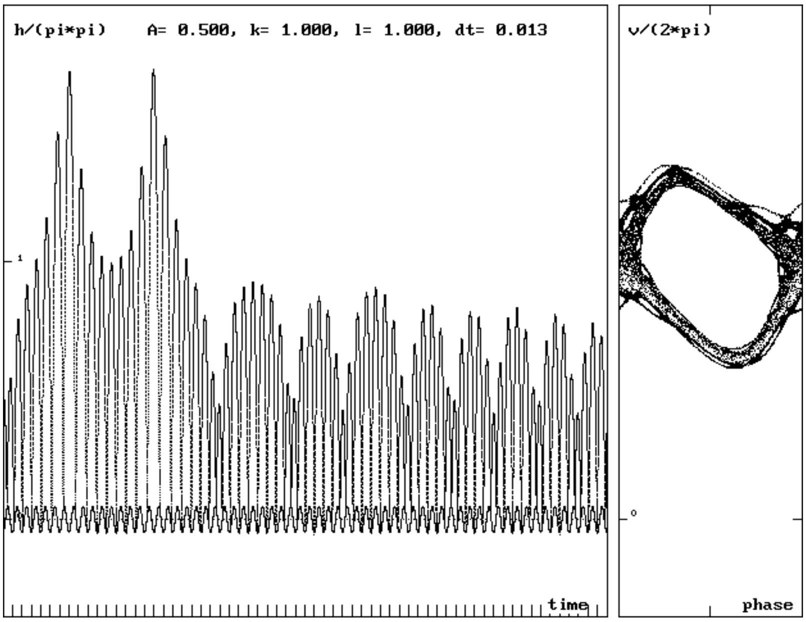

Fig. 43 Stochastic layer spanned on the hyperbolic fixed point. The height-time plot presents just the last, tiny piece of the long trajectory whose trace within the Poincare map covered the complex path.

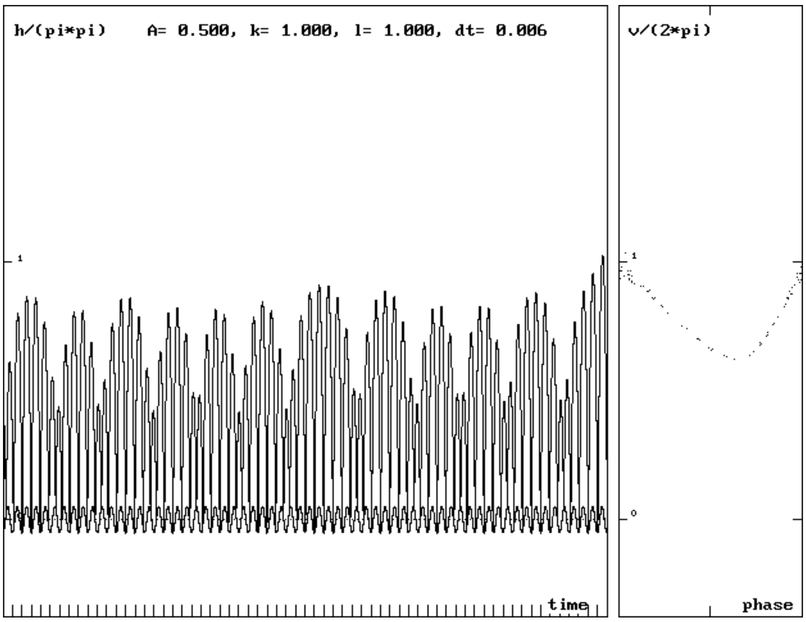

Fig. 44 This piece of the chaotic trajectory looks quite regular at the first sight. A close look at the details reveals, however, that each jump is different. No wonder...

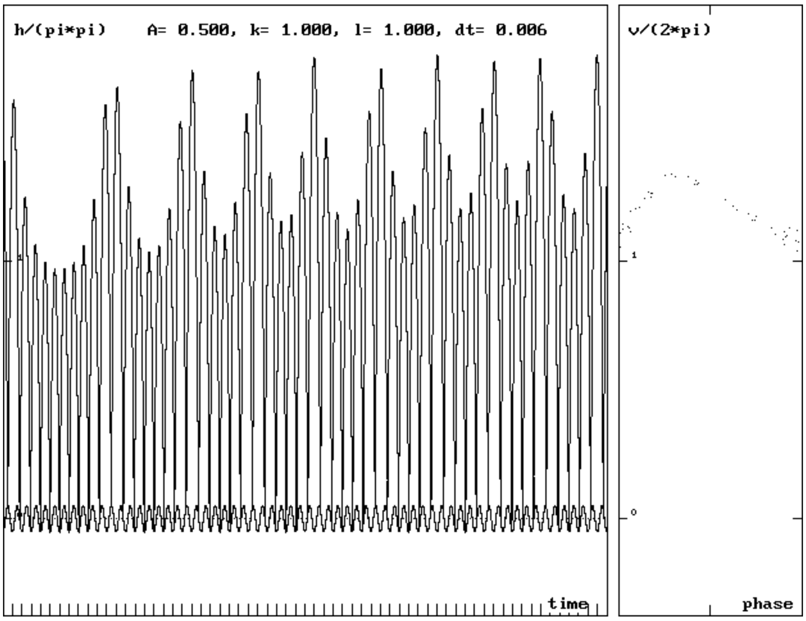

Fig. 45 ...that in a while it starts behaving in a different manner. The "low" and "high" pieces of varying lengths glued together. To predict...

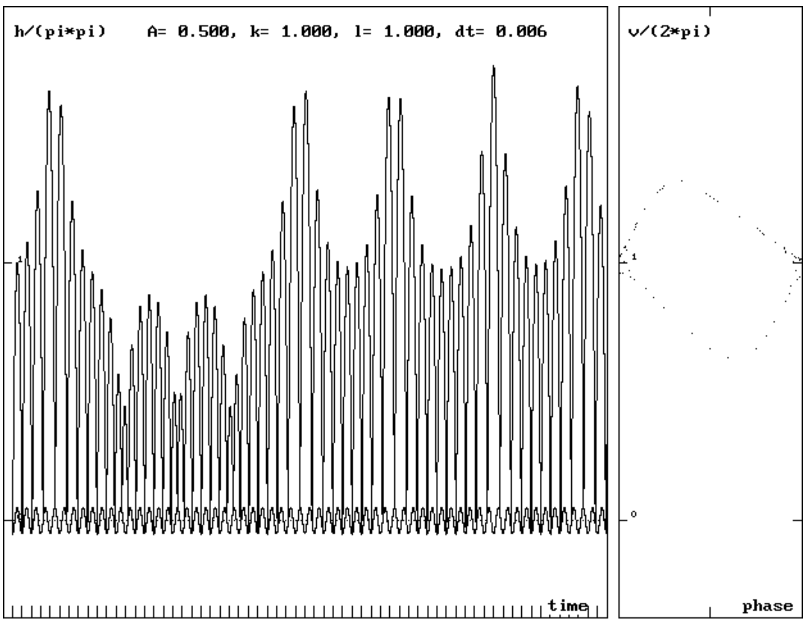

Fig. 46 when a "low" piece will turn into a "high" piece is a hopeless task.

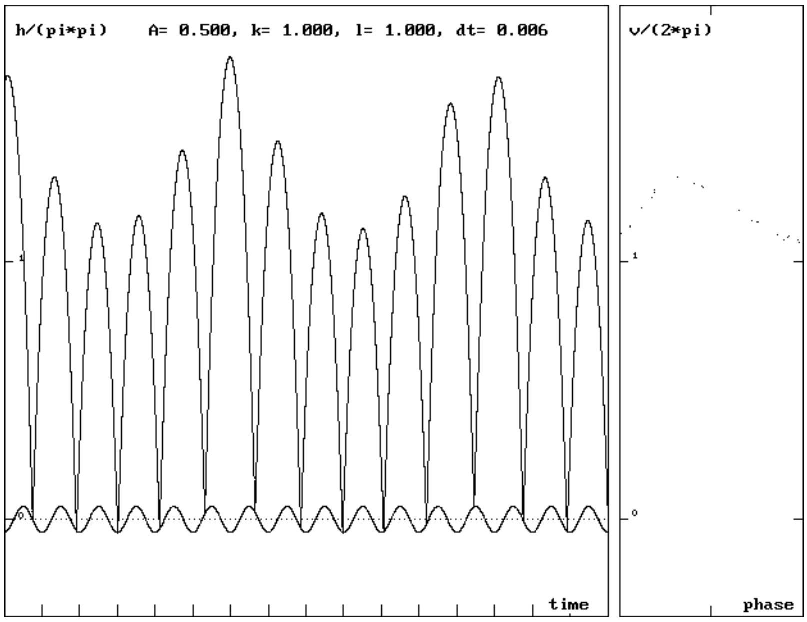

Fig. 47 Two of the elementary units from which the "high" pieces of the stochastic trajectory are built. Note the way in which each of the units swallows one period of the surface vibration.

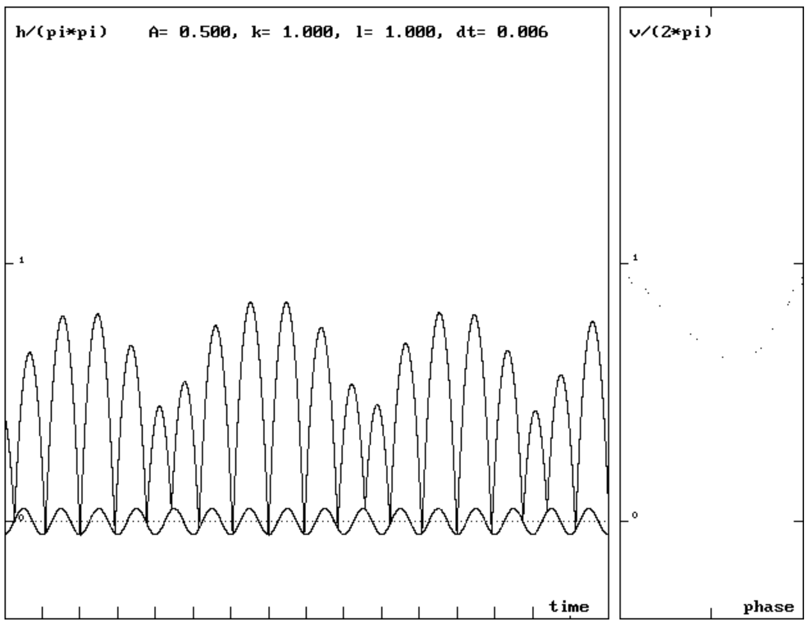

Fig. 48 Units, from which the "low" parts of the stochastic trajectory are built. This time, in each of the units, the number of jumps is by one larger than the number of surface vibration periods.

The stochastic trajectory can be seen as built from units of two kinds. Let us denote them by: *KINK*(+2π) and *KINK*(-2π).

Kinks of one kind are similar but not identical to each other. As explained in the Fig.s above, within any (+2π)KINK the number of the ball's bounces is by one smaller than the number of surface vibration periods beneath; within any of the (-2π)KINK - the other way round. Thus, the whole trajectory can be seen as a random sequence of +2π and -2π phase defects of a perfect trajectory represented by the hyperbolic point itself.[1]

Similar structure of (i) elliptic fixed points surrounded by smooth closed trajectories and (ii) hyperbolic fixed points submerged in layers of stochastic motion are found at many places within the Poincare map. For small values of *A*, layers of the stochastic motion seem to be limited in their thickness and separated from each other. When, however, *A* increases the invisible barriers which separate stochastic layers from each other are broken and the layers become connected. Above a certain critical value of *A*, the last barrier starts leaking and all stochastic layers become connected.

PROBLEMS

1. Following a similar procedure find within the Poincare map some other cycles of fixed points.
2. Determine the kinds of the ball trajectories the cycles represent.
3. Find the stochastic layers within which the hyperbolic fixed points are submerged.
4. Find the critical value of *A* above which all stochastic layers become connected.

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[1] Physical properties of the phase defects are much easier to grasp within the discrete sine-Gordon chain, a many body one dimensional system whose equilibrium configurations remain in a complete analogy with trajectories of the bouncing ball. Studying the fascinating object, one can find out that phase defects of any of its commensurate structures (e.g the simplest one, in which all pendula are hanging down the gravitational field) are not free to move along the chain, and thus within an infinite chain an infinite number of them can coexist. The only problem they encounter is not to stay to close to each other. *A commensurate structure of the sine-Gordon chain saturated with such defects represents a stochastic trajectory of the conservative bouncing ball.* Analogies of this sort can be found for other types of the conservative bouncing ball types of motion.

VI. INELASTIC BOUNCING BALL [2]

At the other end of the interval of the physically reasonable values of the restitution parameter *k*, we find a ball which is completely inelastic i.e. at any collision it adopts its own velocity to that of the surface (but it does not stick to it !). Such a collision can be seen as the ball's endeavour to sit on the surface. There are two possible outcomes of the event:

(i) when such a try ends successfully, the ball remains sitting on the surface until the varying acceleration of the latter will exceed gravity;

(ii) it may happen, however, that although the collision is completely inelastic, the ball fails to sit on the surface. Such an unsuccessful try occurs when the varying acceleration of the surface makes the latter escape from the ball.

Below we present a few examples.

<u>EXPERIMENT</u> 15 *From the "sitting" to the "bouncing" mute mode*

1. Set the restitution factor *k* to 0.
2. Set the surface vibration amplitude *A* to 0.
3. Make *A* to increase slowly and observe motion of the ball.

It is obvious that at k=0 and A=0 the ball stays motionless. As the surface starts moving, the ball starts moving with it until...(see Fig.   )

[2] This chapter has been born after a visit of one of us (P.P) to the Laboratory of Acoustics and Optics of the Condensed Matter at the University of Pierre and Marie Curie, 4 Place Jussieu, Paris. We are grateful to E.Clement, J.Duran and J.Rajchenbach for drowing our attention to the k=0 case, which before this visit seemed always non-physical to us. Experiments which we have been shown convinced us that we were completely wrong.

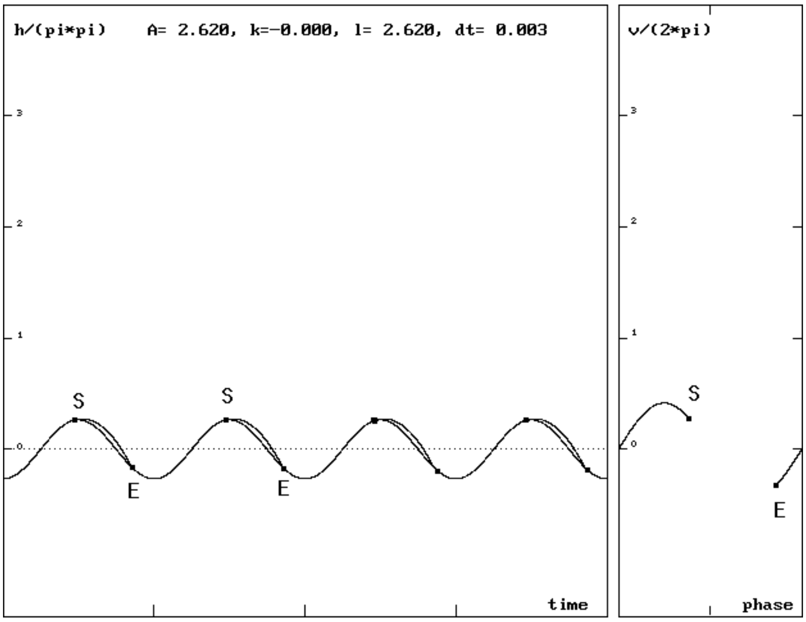

Fig. 49 ...the acceleration of the surface becomes too strong: the ball is forced to leave the surface. Due to complete in elasticity the ball returns to the surface after but a single jump and remains on it until the next moment of the critical acceleration arrives. As the amplitude of the surface motion increases, the time interval within which the ball stays moving with the surface becomes shorter and shorter...

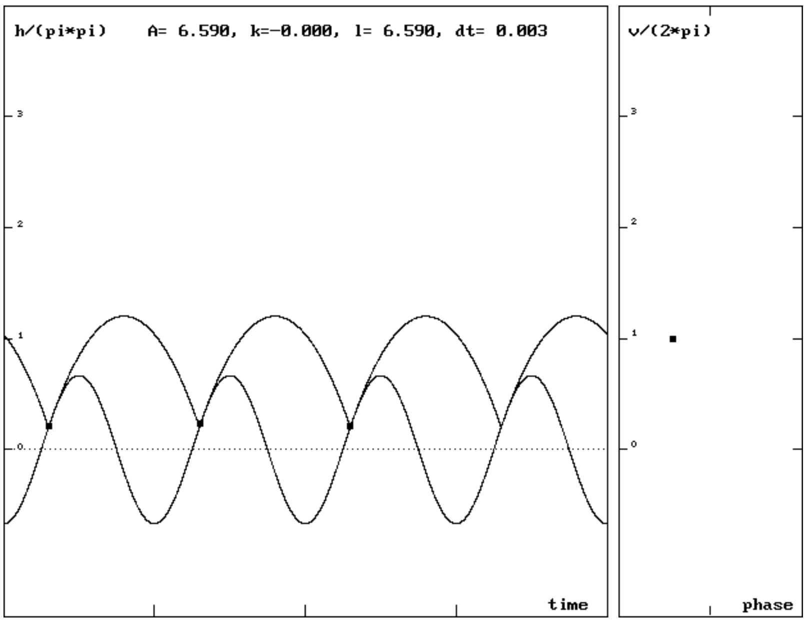

Fig. 50 ... to be reduced eventually but to a single point. The mode of motion reminds very much the $M^{(1)}$ mode.

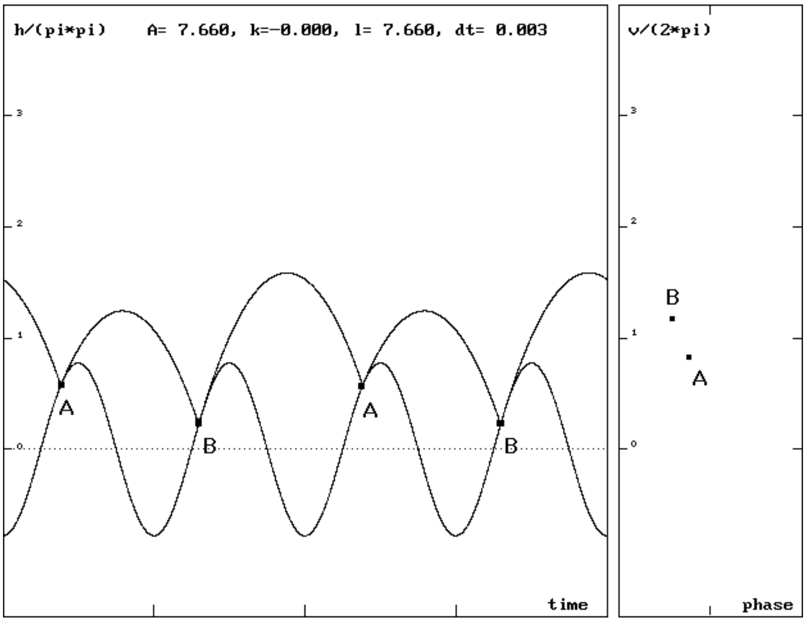

Fig. 51 and it behaves in a similar manner. For instance, it bifurcates into its period doubled version.

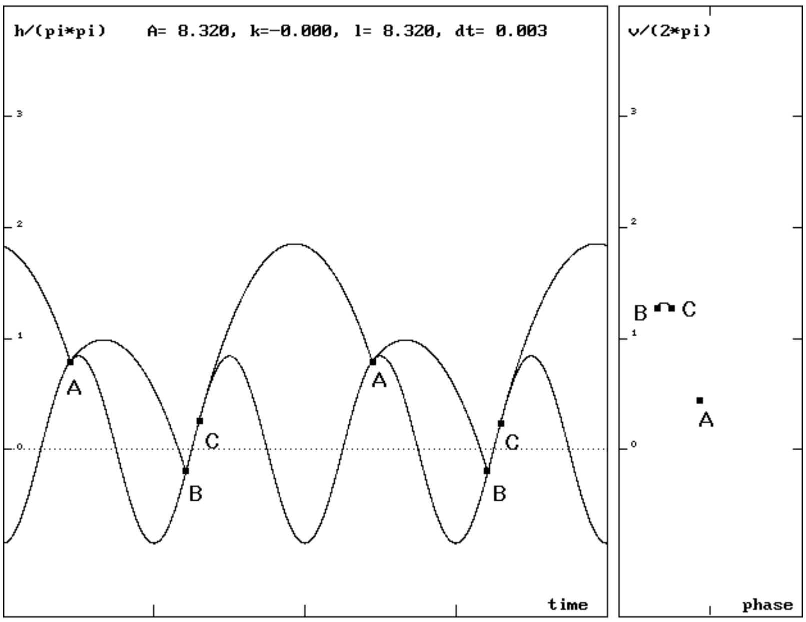

Fig. 52 As *A* increases still further, one of the collision points enters the region where the acceleration is too low what allows the ball to sit for a while on the surface.

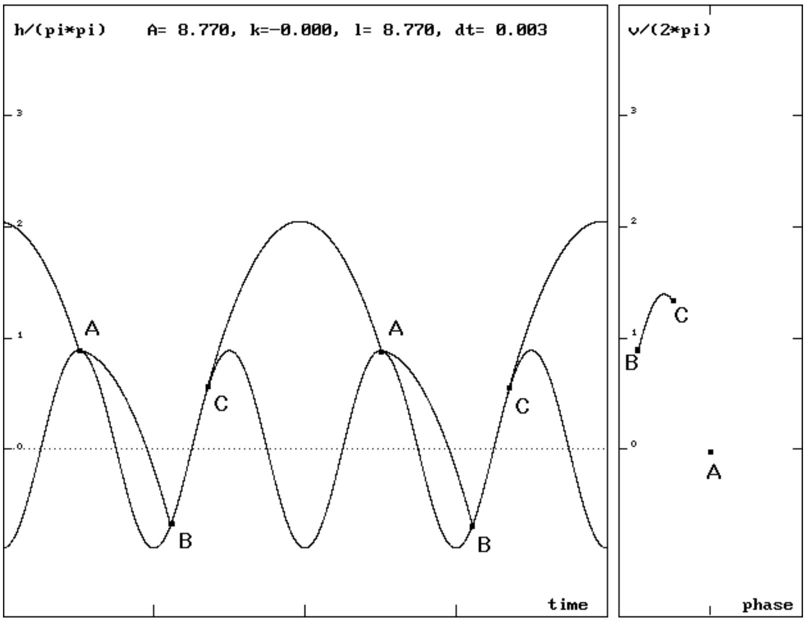

Fig. 53 At a particular value of *A*, one of the collision hits exactly the upper turning point of the surface motion. As a result, the ball enters the following jump with zero velocity.

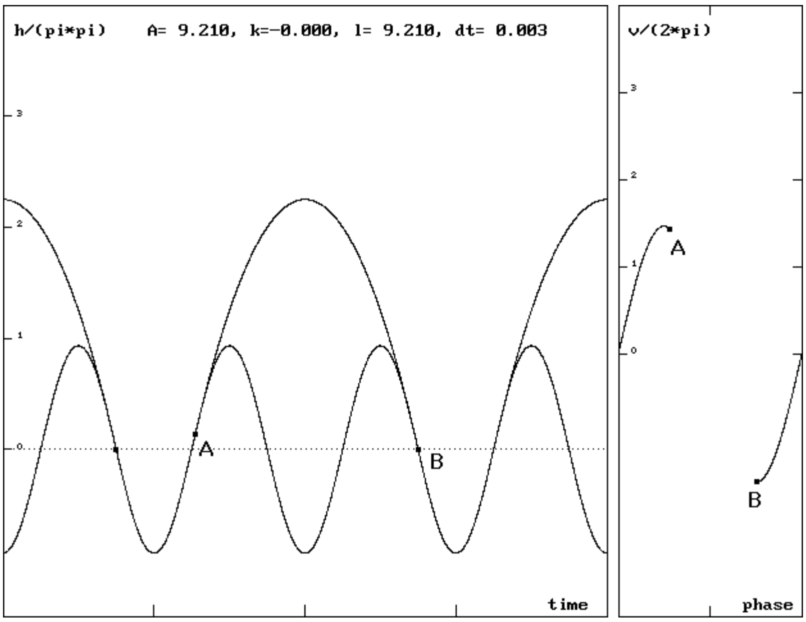

Fig. 54 Eventually, the parabolic trajectory of the first jump lands smoothly on the negative slope of the surface motion. As a result, the mode becomes soundless and no energy is dissipated!

APPENDIX A *A brief description of the software.*

The Bouncing Ball Workbench is a simulation program written in Turbo Pascal 5.0, Borland. It needs an IBM PC with a VGA graphic card and a math-coprocessor. (Without the coprocessor the simulation processes are much slower.)
Below we list all keys which control parameters of the simulated systems, in particular:

- the amplitude *A* with which the collision surface vibrates,
- the restitution factor *k* which determines the part of kinetic energy dissipated during each collision,
- time, height and velocity scales of all plots drawn by the program simultaneously with the simulation processes,
- time step within the integration procedures,

The choice of the keys is compatible with the standard QWERTY type keyboard!

Keys A and S control the surface vibration amplitude *A*.
Keys k and l control the restitution factor *k*.

Keys R and T control the time *dt* step used by the simulation procedure

Keys C and V control the units of the velocity scale of the Poincare map
Key P clears the Poincare map.

Keys Q and W change the time scale of the height-time plots (N-view)
Keys Z and X change the height scale of the height-time plots (N-view)

Keys 1,...,5 drop the ball from different heights.
Key 9 puts the ball onto the moving surface - the easiest way to enter the mute mode.

Key N changes the view from "B" to "N"
Key B changes the view from "N" to "B"

Key . (full stop) freezes the system (N-view) - convenient in situation when a particular trajectory needs to be analysed in detail.
Key E - EXIT

ACKNOWLEDGEMENTS

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